

A FUNCTIONAL APPROACH TOWARDS EIGENVALUE PROBLEMS ASSOCIATED WITH INCOMPRESSIBLE FLOW

SHUANG LIU*

Institute for Mathematical Sciences, Renmin University of China
Beijing 100872, China

YUAN LOU

Department of Mathematics, Ohio State University
Columbus, OH 43210, USA

ABSTRACT. We propose a certain functional which is associated with principal eigenfunctions of the elliptic operator $L_A = -\operatorname{div}(a(x)\nabla) + A\mathbf{V} \cdot \nabla + c(x)$ and its adjoint operator for general incompressible flow $\mathbf{V}$. The functional can be applied to establish the monotonicity of the principal eigenvalue $\lambda_1(A)$, as a function of the advection amplitude A , for the operator L_A subject to Dirichlet, Robin and Neumann boundary conditions. This gives a new proof of a conjecture raised by Berestycki, Hamel and Nadirashvili [5]. The functional can also be used to prove the monotonicity of the normalized speed $c^*(A)/A$ for general incompressible flow, where $c^*(A)$ is the minimal speed of traveling fronts. This extends an earlier result of Berestycki [3] for steady shear flow.

1. Introduction. There have been extensive studies on the reaction-diffusion equations of the form

$$w_t = \operatorname{div}(a(x)\nabla w) - A\mathbf{V} \cdot \nabla w + wf(x, w), \quad (1)$$

which model various physical, chemical, and biological processes: On unbounded domains [17, 44], compact manifolds [10], and bounded domains with appropriate boundary conditions [1, 5, 7, 34]. Function w represents the density of a population or a substance diffusing with diffusion matrix $a(x)$, reacting through the nonlinearity wf , and advected by the stationary fluid flow $\mathbf{V}$ in heterogeneous media.

Let Ω be a bounded region of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $\mathbf{n}(x)$ be the outward unit normal vector at $x \in \partial\Omega$. Consider equation (1) defined on Ω and suppose that w satisfies $bw + (1-b)[a(x)\nabla w] \cdot \mathbf{n} = 0$ on $\partial\Omega$ with parameter $b \in [0, 1]$. The stability of steady state $w \equiv 0$ and the minimal speed $c^*(A)$ of traveling fronts for equation (1) are associated with the principal eigenvalue, denoted as $\lambda_1(A)$, for the linear eigenvalue problem

$$L_A u := -\operatorname{div}(a(x)\nabla u) + A\mathbf{V} \cdot \nabla u + c(x)u = \lambda_1(A)u,$$

subject to appropriate boundary conditions, where $c(x) = -f(x, 0)$.

2010 *Mathematics Subject Classification.* Primary: 35J20, 35J60, 35P15.

Key words and phrases. Principal eigenvalue, monotonicity, incompressible flow, min-max characterization, minimal speed.

SL was partially supported by the NSFC grant No. 11571364 and the Outstanding Innovative Talents Cultivation Funded Programs 2018 of Renmin University of China. YL was partially supported by the NSF grants DMS-1411176 and DMS-1853561.

*Corresponding author: Shuang Liu.

Of particular interest is the dependence of the principal eigenvalue $\lambda_1(A)$ and the minimal speed $c^*(A)$ on the advection amplitude A . Let us focus on the case where vector field $\mathbf{V}$ is divergence free, i.e., $\operatorname{div} \mathbf{V} = 0$ in Ω , while the case of gradient flow $\mathbf{V} = \nabla m$ for some $m \in C^2(\bar{\Omega})$ has been investigated by Chen and Lou in [8] and [9]. The purpose of this paper is to introduce a certain functional to prove the monotonicity of $\lambda_1(A)$ and $c^*(A)/A$ with respect to A .

Set $L_A^* := -\operatorname{div}(a(x)\nabla) - A\mathbf{V} \cdot \nabla + c(x)$ as the adjoint operator of L_A . By u_A, v_A we further denote the principal eigenfunctions corresponding to L_A and L_A^* with appropriate boundary conditions, respectively, which are normalized by $\int_{\Omega} u_A^2 dx = \int_{\Omega} u_A v_A dx = 1$. In terms of operator L_A and u_A, v_A , we now introduce functional J_A by

$$J_A(\omega) = \int_{\Omega} u_A v_A \left(\frac{L_A \omega}{\omega} \right) dx, \quad (2)$$

which is defined on some cone $\mathbb{S}$. Such a functional turns out to be new, which is different from the general relative entropy introduced in [30]. A direct observation from the definition of functional J_A leads to $J_A(u_A) = \lambda_1(A)$ and a far less obvious result (see Lemma 2.2) says that functional J_A attains its maximum at the principal eigenfunction u_A and its scalar multiples. This is crucial in the proof of the monotonicity of $\lambda_1(A)$ and $c^*(A)/A$ and it also allows us to explore a new min-max characterization of $\lambda_1(A)$.

1.1. Monotonicity and boundedness of $\lambda_1(A)$. Firstly, we focus on the following eigenvalue problem subject to general boundary conditions:

$$\begin{cases} L_A u = -\operatorname{div}(a(x)\nabla u) + A\mathbf{V} \cdot \nabla u + c(x)u = \lambda_1(A)u & \text{in } \Omega, \\ bu + (1-b)[a(x)\nabla u] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (3)$$

Assume that $c \in C^\alpha(\bar{\Omega})$ and the diffusion matrix $a(x)$ is symmetric and uniformly elliptic $C^{1,\alpha}(\bar{\Omega})$ matrix field satisfying

$$\exists 0 < \gamma_1 < \gamma_2, \text{ such that } \gamma_1 |\xi|^2 \leq \xi^T a(x) \xi \leq \gamma_2 |\xi|^2, \forall x \in \Omega, \forall \xi \in \mathbb{R}^N,$$

for some constant $\alpha \in (0, 1)$. Furthermore, we always assume that the vector field $\mathbf{V} \in C^1(\bar{\Omega})$ satisfying $\operatorname{div} \mathbf{V} = 0$ in Ω , whereas an additional assumption stating that $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$ is assumed for the case of $0 \leq b < 1$. Under these assumptions the Krein-Rutman Theorem guarantees the existence of the principal eigenvalue $\lambda_1(A)$ and it can be easily shown that $\lambda_1(A)$ is symmetric in A . Therefore, throughout this paper we shall assume $A \geq 0$.

For such flow $\mathbf{V}$, Berestycki et al. investigated in [5] the asymptotic behavior of $\lambda_1(A)$ as A approaches infinity, and they identified a direct link between the limit of $\lambda_1(A)$ and the first integral set of $\mathbf{V}$, defined as

$$\mathcal{I}_b = \begin{cases} \{\varphi \in H^1(\Omega) : \varphi \neq 0, \mathbf{V} \cdot \nabla \varphi = 0 \text{ a.e. in } \Omega\}, & 0 \leq b < 1, \\ \{\varphi \in H_0^1(\Omega) : \varphi \neq 0, \mathbf{V} \cdot \nabla \varphi = 0 \text{ a.e. in } \Omega\}, & b = 1. \end{cases}$$

More precisely, Berestycki et al. showed in [5] that for the operator L_A defined on Ω with Dirichlet ($b = 1$) or Neumann ($b = 0$) boundary conditions, $\lambda_1(A)$ stays bounded as $A \rightarrow +\infty$ if and only if $\mathcal{I}_1 \neq \emptyset$ or $\mathcal{I}_0 \neq \emptyset$, respectively. Furthermore, they proved that for any $A \geq 0$,

$$\lambda_1(0) \leq \lambda_1(A) \leq \lim_{A \rightarrow +\infty} \lambda_1(A) = \inf_{\omega \in \mathcal{I}_0 \text{ or } \mathcal{I}_1} \frac{\int_{\Omega} \nabla \omega \cdot [a(x)\nabla \omega] dx + \int_{\Omega} c(x)\omega^2 dx}{\int_{\Omega} \omega^2 dx}. \quad (4)$$

That is, $\lambda_1(A)$ attains its minimum at $A = 0$ and its maximum at $A = \infty$. As mentioned in [5], $\lambda_1(A)$ is a nondecreasing function of $|A|$ if $\mathbf{V}$ is an incompressible gradient flow. One of the goals of this paper is to give a new proof of the following result.

Theorem 1.1. *Let $\operatorname{div} \mathbf{V} = 0$ in Ω and additionally $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$ for $0 \leq b < 1$. Then $\lambda_1(A)$ is non-decreasing for $A \geq 0$. Furthermore,*

- (i) *If $u_0 \notin \mathcal{I}_b$, then $\frac{\partial \lambda_1}{\partial A}(A) > 0$ for every $A > 0$;*
- (ii) *If $u_0 \in \mathcal{I}_b$, then $\lambda_1(A) \equiv \lambda_1(0)$ for every $A > 0$.*

Here u_0 is the principal eigenfunction of L_0 satisfying

$$\begin{cases} -\operatorname{div}(a(x)\nabla u_0) + c(x)u_0 = \lambda_1(0)u_0 & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ bu_0 + (1-b)[a(x)\nabla u_0] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof of Theorem 1.1 was first given by Godoy et al. [19] via a variant of the min-max formula derived in [18] for principal eigenvalues. Our proof relies heavily on properties of functional J_A defined in (2) by identifying the definition cone $\mathbb{S}$ as

$$\mathbb{S}_b = \begin{cases} \{\varphi \in C^2(\Omega) \cap C^1(\bar{\Omega}) : \varphi > 0 \text{ in } \Omega, \ b\varphi + (1-b)[a(x)\nabla\varphi] \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, & \text{for } 0 \leq b < 1, \\ \{\varphi \in C^2(\Omega) \cap C^1(\bar{\Omega}) : \varphi > 0 \text{ in } \Omega, \ \varphi = 0 \text{ on } \partial\Omega, \ \nabla\varphi \cdot \mathbf{n} < 0 \text{ on } \partial\Omega\}, & \text{for } b = 1. \end{cases}$$

Our proof avoids the min-max formula of principal eigenvalues for non-symmetric operators and gives an explicit expression for the derivatives of $\lambda_1(A)$.

Theorem 1.1 implies that the strict monotonicity of $\lambda_1(A)$ with respect to the advection amplitude A relies on u_0 , the principal eigenfunction of operator L_0 . Interpreting this in the context of convection-enhanced diffusion, Theorem 1.1 suggests that larger advection amplitude generally produces faster mixing for reaction-diffusion-advection equation (1) as long as $u_0 \notin \mathcal{I}_b$. In this sense, Theorem 1.1 seems to refine the well-known statement that mixing by an incompressible flow enhances diffusion in various contexts [10, 17, 20, 21, 26, 27, 40, 44].

Our next result, as a corollary of Theorem 1.1, provides the boundedness and asymptotic behavior of $\lambda_1(A)$ for Robin boundary conditions, consistent with the main result in [5] for Neumann boundary conditions.

Theorem 1.2. *If $0 \leq b < 1$, the limit $\lim_{A \rightarrow +\infty} \lambda_1(A)$ always exists, is finite and satisfies*

$$\lim_{A \rightarrow +\infty} \lambda_1(A) \leq \inf_{\omega \in \mathcal{I}_b} \frac{\frac{b}{1-b} \int_{\partial\Omega} \omega^2 dS_x + \int_{\Omega} \nabla\omega \cdot [a(x)\nabla\omega] dx + \int_{\Omega} c(x)\omega^2 dx}{\int_{\Omega} \omega^2 dx}.$$

In particular, the principal eigenvalue $\lambda_1(A)$ of (3) with $0 \leq b < 1$, is uniformly bounded.

The proof of the boundedness for $\lambda_1(A)$ in Theorem 1.2 is essentially due to Berestycki et al. [5]. Nevertheless, the existence of the limit $\lim_{A \rightarrow \infty} \lambda_1(A)$ for Robin boundary conditions appears to be new.

1.2. A new min-max characterization of $\lambda_1(A)$. The characterization of the principal eigenvalue has always been an interesting and active topic, and we refer to Donsker and Varadhan, Nussbaum and Pinchover for some earlier works [13, 15, 37]. Employing the maximum principle, Protter and Weinberger [38] established a classical characterization of the principal eigenvalue $\lambda_1(A)$ given by the min-max formula

$$\lambda_1(A) = \sup_{\omega \in \mathbb{S}_b} \inf_{x \in \Omega} \left[\frac{L_A \omega(x)}{\omega(x)} \right]. \quad (5)$$

This characterization is valid for general elliptic operators in both bounded and unbounded domains [37, 38]. As a byproduct of properties of functional J_A , we have the following characterization for $\lambda_1(A)$:

Theorem 1.3. *The principal eigenvalue $\lambda_1(A)$ of problem (3), with $\operatorname{div} \mathbf{V} = 0$ in Ω and additionally $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$ for $0 \leq b < 1$, can be characterized as*

$$\lambda_1(A) = \inf_{p \in L^2(\Omega), \int_{\Omega} p^2 = 1} \sup_{\omega \in \mathbb{S}_b} \int_{\Omega} p^2(x) \left(\frac{L_A \omega}{\omega} \right) dx. \quad (6)$$

This min-max formula may not be valid for general second elliptic operators, and it reduces to the classical Rayleigh-Ritz formula when $\mathbf{V} = 0$, by treating $p^2 dx$ as some probability measure; See Remark 2 for details. Different from formula (5), the min-max characterization in Theorem 1.3 relies on the properties of functional J_A . They however may be connected via a min-max theorem in [39]. Via functional J_A we observe that the min-max formula attains the extremum when $p^2 = u_A v_A$.

1.3. Monotonicity of $c^*(A)/A$. Another application of functional J_A is concerned with the minimal speed $c^*(A)$ of traveling fronts for equation (1) with Neumann boundary condition ($b = 0$). As in [4, 6, 40], we here consider the general periodic setting described in Subsection 5.1. One interesting question is the dependence of the minimal speed $c^*(A)$ on amplitude A .

From the physical point of view, the presence of incompressible flow $\mathbf{V}$ in equation (1) generally improves mixing [22, 23, 40] and is thus expected to enhance $c^*(A)$. Difficulties however may arise because of the interplay between the stream lines of general incompressible flow $\mathbf{V}$; See [29, 40]. Some results focused on the case where $\mathbf{V} = (\alpha(y), 0, \dots, 0)$, so-called shear flow in a straight cylinder. Examples are known for which the minimal speed $c^*(A)$ is asymptotically linear with respect to A in the sense of $c^*(A)/A \rightarrow \rho > 0$ as $A \rightarrow +\infty$ [3, 22, 23], while $c^*(A)/A \rightarrow 0$ could happen in general incompressible flow. We refer to [44] for the precise limit as $A \rightarrow +\infty$. Furthermore, $c^*(A)$ is increasing in A , $c^*(A)/A$ is decreasing in A for shear flow [3, 33]. The monotonicity of $c^*(A)$ and $c^*(A)/A$ however remain open for general incompressible flow $\mathbf{V}$; See Remark 1.9 in [6] and Remark 1.6 in [22] for details. By functional J_A , we can prove the monotonicity of $c^*(A)/A$ for general incompressible flow.

Theorem 1.4. *Suppose $\operatorname{div} \mathbf{V} = 0$ in Ω and additionally $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Then $c^*(A)/A$ is strictly decreasing in amplitude $A > 0$.*

Theorem 1.4 extends the result proved by Berestycki [3] and Nadin [33] for shear flow. For shear flow, we can write L_A as a symmetric operator by some manipulations. Theorem 1.4 henceforth can be proved by the Rayleigh-Ritz formula of the principal eigenvalues. However, this technique does not appear to work for general incompressible flow $\mathbf{V}$. Heinze introduced in [23] an interesting change of variables

to prove the monotonicity of $c^*(A)/A$. Such change of variables then was applied by Nadin [32] to give a new characterization of principal eigenvalue for a nonsymmetric operator. Different from Heinze's argument, our proof of Theorem 1.4 relies heavily on functional J_A defined by (2), which allows us to obtain an explicit expression (32) for the derivatives of $c^*(A)/A$. Unlike the framework of bounded domain, some modifications for the definition cone $\mathbb{S}$ are required for the periodic setting here.

The rest of this paper is organized as follows: In Section 2, we shall give some properties of functional J_A on bounded domain. Section 3 is devoted to the proof of Theorems 1.1 and 1.2. In Section 4 we establish the new min-max characterization of $\lambda_1(A)$. We then use functional J_A to prove the monotonicity of $c^*(A)/A$ in Sections 5 and 6 under the general periodic setting. Finally, the implications of our method/results and some open questions will be discussed in Section 7.

2. Functional J_A on bounded domain. We shall present some properties of functional J_A defined by (2) on $\mathbb{S}_b$ in this section. Before proceeding further, we point out again that throughout this paper, u_A and v_A normalized by $\int_{\Omega} u_A^2 dx = \int_{\Omega} u_A v_A dx = 1$ are the principal eigenfunctions corresponding to L_A and L_A^* , respectively, with general boundary conditions. Precisely, $u_A > 0$ in Ω satisfies (3) and v_A satisfies

$$\begin{cases} L_A^* v_A = -\operatorname{div}(a(x)\nabla v_A) - A\mathbf{V} \cdot \nabla v_A + c(x)v_A = \lambda_1(A)v_A & \text{in } \Omega, \\ v_A > 0 & \text{in } \Omega, \\ bv_A + (1-b)[a(x)\nabla v_A] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Due to the slight difference between the definitions of functional J_A in the cases of $0 \leq b < 1$ and $b = 1$, we divide this section into two subsections.

2.1. Neumann and Robin boundary conditions: $0 \leq b < 1$. Recalling the regularity requirements of coefficients c , $\mathbf{V}$ and matrix field $a(x)$, Sobolev embedding theorem implies that $u_A, v_A \in C^{2,\alpha}(\Omega)$ and $u_A, v_A \in \mathbb{S}_b$ for $0 \leq b < 1$. We emphasize here that the constant b is confined to $0 \leq b < 1$ unless otherwise specified, and the incompressible flow $\mathbf{V}$ satisfies $\operatorname{div}\mathbf{V} = 0$ in Ω with $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$ in this subsection. We now recall the functional associated to operator L_A with Neumann or Robin boundary conditions, defined on $\mathbb{S}_b$ as in Section 1,

$$J_A(\omega) = \int_{\Omega} u_A v_A \left(\frac{L_A \omega}{\omega} \right) dx, \quad \omega \in \mathbb{S}_b.$$

Define $\tilde{\mathbb{S}}_b := \left\{ \varphi \in C^2(\Omega) \cap C^1(\bar{\Omega}) : b\varphi + (1-b)[a(x)\nabla\varphi] \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\}$. By definition of J_A , we show that principal eigenfunction u_A is a critical point of J_A .

Lemma 2.1. $J'_A(u_A)\varphi = 0$ for all $\varphi \in \tilde{\mathbb{S}}_b$.

Proof. For any $\varphi \in \tilde{\mathbb{S}}_b$, the Fréchet derivation $J'_A(\omega)$ can be written as

$$J'_A(\omega)\varphi = \int_{\Omega} u_A v_A \left(\frac{L_A \varphi}{\omega} - \frac{\varphi L_A \omega}{\omega^2} \right) dx.$$

In view of $\varphi \in \tilde{\mathbb{S}}_b$, direct calculation gives

$$J'_A(u_A)\varphi = \int_{\Omega} u_A v_A \left(\frac{L_A \varphi}{u_A} - \frac{\varphi L_A u_A}{u_A^2} \right) dx$$

$$\begin{aligned}
&= \int_{\Omega} v_A (L_A \varphi - \lambda_1(A) \varphi) dx \\
&= \int_{\Omega} \varphi L_A^* v_A dx - \lambda_1(A) \int_{\Omega} v_A \varphi dx \\
&= 0,
\end{aligned}$$

which completes the proof of Lemma 2.1. $\square$

Next we establish a crucial property of functional J_A .

Proposition 2.2. *For any $\omega \in \mathbb{S}_b$, the following formula holds:*

$$J_A(u_A) = J_A(\omega) + \int_{\Omega} u_A v_A \left\{ \left[\nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} dx.$$

Proof. First, for any $\omega \in \mathbb{S}_b$, a simple but useful observation leads to

$$\begin{aligned}
&J_A(\omega) \\
&= - \int_{\Omega} u_A v_A \left[\frac{\operatorname{div}(a(x) \nabla \omega)}{\omega} \right] dx + A \int_{\Omega} u_A v_A \left[\frac{\mathbf{V} \cdot \nabla \omega}{\omega} \right] dx + \int_{\Omega} u_A v_A c dx \\
&= - \int_{\partial \Omega} u_A v_A [a(x) \nabla \log \omega] \cdot \mathbf{n} dS_x + \int_{\Omega} \nabla \left(\frac{u_A v_A}{\omega} \right) \cdot [a(x) \nabla \omega] dx \\
&\quad + A \int_{\Omega} u_A v_A \mathbf{V} \cdot \nabla \log \omega dx + \int_{\Omega} u_A v_A c dx \\
&= - \int_{\partial \Omega} u_A v_A [a(x) \nabla \log \omega] \cdot \mathbf{n} dS_x - \int_{\Omega} u_A v_A \left\{ (\nabla \log \omega) \cdot [a(x) \nabla \log \omega] \right\} dx \\
&\quad + \int_{\Omega} [A u_A v_A \mathbf{V} + a(x) \nabla (u_A v_A)] \cdot \nabla \log \omega dx + \int_{\Omega} u_A v_A c dx.
\end{aligned} \tag{7}$$

Using equality (7), the Fréchet derivation $J'_A(u_A)$ can be rewritten as

$$\begin{aligned}
&J'_A(u_A) \varphi \\
&= - \int_{\partial \Omega} u_A v_A \left[a(x) \nabla \left(\frac{\varphi}{u_A} \right) \right] \cdot \mathbf{n} dS_x \\
&\quad + \int_{\Omega} [A u_A v_A \mathbf{V} + a(x) \nabla (u_A v_A)] \cdot \nabla \left(\frac{\varphi}{u_A} \right) dx \\
&\quad - 2 \int_{\Omega} u_A v_A \left\{ (\nabla \log u_A) \cdot \left[a(x) \nabla \left(\frac{\varphi}{u_A} \right) \right] \right\} dx,
\end{aligned} \tag{8}$$

for all $\varphi \in \tilde{\mathbb{S}}_b$.

To prove Proposition 2.2, some elementary but a bit tedious manipulations are needed. Together with equality (7), direct calculation yields

$$\begin{aligned}
&J_A(u_A) - J_A(\omega) \\
&= \int_{\partial \Omega} u_A v_A \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \mathbf{n} dS_x + \int_{\Omega} u_A v_A \left\{ (\nabla \log \omega) \cdot [a(x) \nabla \log \omega] \right\} dx \\
&\quad - \int_{\Omega} u_A v_A \left\{ (\nabla \log u_A) \cdot [a(x) \nabla \log u_A] \right\} dx \\
&\quad - \int_{\Omega} [A u_A v_A \mathbf{V} + a(x) \nabla (u_A v_A)] \cdot \nabla \log \left(\frac{\omega}{u_A} \right) dx
\end{aligned}$$

$$\begin{aligned}
&= \int_{\partial\Omega} u_A v_A \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \mathbf{n} \, dS_x \\
&\quad + \int_{\Omega} u_A v_A \left\{ \left[\nabla \log (u_A \omega) \right] \cdot \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} dx \\
&\quad - \int_{\Omega} \left[A u_A v_A \mathbf{V} + a(x) \nabla (u_A v_A) \right] \cdot \nabla \log \left(\frac{\omega}{u_A} \right) dx \\
&= \int_{\Omega} u_A v_A \left\{ \left[\nabla \log \left(\frac{\omega}{u_A} \right) \right] dx + 2 \nabla \log u_A \cdot \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} dx \\
&\quad + \int_{\partial\Omega} u_A v_A \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \mathbf{n} \, dS_x \\
&\quad - \int_{\Omega} \left[A u_A v_A \mathbf{V} + a(x) \nabla (u_A v_A) \right] \cdot \nabla \log \left(\frac{\omega}{u_A} \right) dx \\
&= \int_{\Omega} u_A v_A \left\{ \left[\nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} dx \\
&\quad + \int_{\partial\Omega} u_A v_A \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \mathbf{n} \, dS_x \\
&\quad + 2 \int_{\Omega} u_A v_A \left\{ (\nabla \log u_A) \cdot \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} dx \\
&\quad - \int_{\Omega} \left[A u_A v_A \mathbf{V} + a(x) \nabla (u_A v_A) \right] \cdot \nabla \log \left(\frac{\omega}{u_A} \right) dx,
\end{aligned}$$

where we have used the symmetry of matrix field $a(x)$ and the boundary conditions of ω and u_A . By straightforward calculations we have $u_A \log \left(\frac{\omega}{u_A} \right) \in \tilde{\mathbb{S}}_b$ for any $\omega \in \mathbb{S}_b$. Choosing $\varphi = u_A \log \left(\frac{\omega}{u_A} \right)$ in equality (8), by Lemma 2.1 we have

$$\begin{aligned}
J_A(u_A) - J_A(\omega) &= \int_{\Omega} u_A v_A \left\{ \left[\nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} dx - J'_A(u_A) \varphi \\
&= \int_{\Omega} u_A v_A \left\{ \left[\nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \left[a(x) \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} dx.
\end{aligned}$$

The assertion of Proposition 2.2 thus follows. $\square$

The following result is an immediate consequence of Proposition 2.2.

Corollary 2.3.

$$\int_{\Omega} v_A L_A u_A dx - \int_{\Omega} u_A L_A v_A dx = \int_{\Omega} u_A v_A \left\{ \left[\nabla \log \left(\frac{v_A}{u_A} \right) \right] \cdot \left[a(x) \nabla \log \left(\frac{v_A}{u_A} \right) \right] \right\} dx.$$

Proof. A simple observation leads to

$$\int_{\Omega} u_A L_A v_A dx = \int_{\Omega} u_A v_A \left(\frac{L_A v_A}{v_A} \right) dx = J_A(v_A),$$

and analogously $\int_{\Omega} v_A L_A u_A dx = J_A(u_A)$. Hence Corollary 2.3 follows from Proposition 2.2. $\square$

2.2. Dirichlet boundary conditions: $b = 1$. The case of Dirichlet boundary conditions is slightly different from the Neumann or Robin boundary conditions, as noted in [5]. It is perhaps worth pointing out that in this case, the functional J_A shall be defined on $\mathbb{S}_1$ and the extra assumption $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$ is not needed for further discussions. Hopf Boundary Lemma implies that $\nabla u_A \cdot \mathbf{n} < 0$ and

$\nabla v_A \cdot \mathbf{n} < 0$ on $\partial\Omega$, and thus $u_A, v_A \in \mathbb{S}_1$ so that $J_A(u_A)$ and $J_A(v_A)$ are well defined. Moreover, the adjoint operator of L_A subject to Dirichlet boundary conditions can be written as $L_A^* = -\operatorname{div}(a(x)\nabla) - A\mathbf{V} \cdot \nabla + c(x)$ without $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$, due to $u_A = 0$ on $\partial\Omega$. Thanks to $\nabla\omega \cdot \mathbf{n} < 0$ on $\partial\Omega$, we have $\frac{u_A v_A}{\omega} = 0$ on $\partial\Omega$ to get $\int_{\partial\Omega} u_A v_A [a(x)\nabla \log \omega] \cdot \mathbf{n} dS_x = 0$ in equality (7).

With the same argument as in the Neumann or Robin boundary conditions, Lemma 2.1 also holds true in this case, i.e., $J'_A(u_A)\varphi = 0$ for all $\varphi \in \tilde{\mathbb{S}}_1$. Based on this fact, the formula in Proposition 2.2 remains true. As the proof is similar, thus it is omitted. Therefore, the properties of functional J_A listed in subsection 2.1 hold for all $0 \leq b \leq 1$.

3. Monotonicity and boundedness of $\lambda_1(A)$. Our goal of this section is to show Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Firstly, if $u_0 \in \mathcal{I}_b$, then for every $A > 0$, u_0 satisfies

$$\begin{cases} -\operatorname{div}(a(x)\nabla u_0) + A\mathbf{V} \cdot \nabla u_0 + c(x)u_0 = \lambda_1(0)u_0 & \text{in } \Omega, \\ u_0 > 0 & \text{in } \Omega, \\ bu_0 + (1-b)[a(x)\nabla u_0] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, $\lambda_1(A) = \lambda_1(0)$ for all $A > 0$. This proves part (i).

For the proof of part (ii), we assume that $u_0 \notin \mathcal{I}_b$. Differentiate equation (3) with respect to A and denote $\frac{\partial u_A}{\partial A} = u'_A$ for the sake of brevity, we obtain

$$\begin{cases} -\operatorname{div}[a(x)\nabla u'_A] + A\mathbf{V} \cdot \nabla u'_A + \mathbf{V} \cdot \nabla u_A + c(x)u'_A = \frac{\partial \lambda_1}{\partial A}(A)u_A + \lambda_1(A)u'_A & \text{in } \Omega, \\ bu'_A + (1-b)[a(x)\nabla u'_A] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \quad \int_{\Omega} u'_A u_A dx = 0. \end{cases} \quad (9)$$

Multiply (9) by v_A and integrate the result in Ω , together with the definition of v_A we have

$$\frac{\partial \lambda_1}{\partial A}(A) = \int_{\Omega} v_A \mathbf{V} \cdot \nabla u_A dx. \quad (10)$$

Observe that $u_0 = v_0$ for $A = 0$. This leads to

$$\frac{\partial \lambda_1}{\partial A}(0) = \frac{1}{2} \int_{\Omega} \mathbf{V} \cdot \nabla u_0^2 dx = 0.$$

Here we used that $\mathbf{V}$ is divergence free together with $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$ for $0 \leq b < 1$ and $u_0 = 0$ on $\partial\Omega$ for $b = 1$.

Claim. For each $A > 0$, $\frac{\partial \lambda_1}{\partial A}(A) \geq 0$, and either $\frac{\partial \lambda_1}{\partial A}(A) > 0$, or $\lambda_1(A) = \lambda_1(0)$.

To establish this assertion, recall the definition of L_A and L_A^* to rewrite equality (10) as

$$\frac{\partial \lambda_1}{\partial A}(A) = \frac{1}{2A} \int_{\Omega} v_A (L_A - L_A^*) u_A dx = \frac{1}{2A} \left[\int_{\Omega} v_A L_A u_A dx - \int_{\Omega} u_A L_A v_A dx \right].$$

A direct application of Corollary 2.3 and positive definiteness of $a(x)$ yields

$$\frac{\partial \lambda_1}{\partial A}(A) = \frac{1}{2} \int_{\Omega} u_A v_A \left\{ \left[\nabla \log \left(\frac{v_A}{u_A} \right) \right] \cdot \left[a(x) \nabla \log \left(\frac{v_A}{u_A} \right) \right] \right\} dx \geq 0,$$

and $\frac{\partial \lambda_1}{\partial A}(A) = 0$ if and only if $u_A = cv_A$ for some $c > 0$. By $\int_{\Omega} u_A^2 = 1$ and $\int_{\Omega} u_A v_A = 1$, we see that $c = 1$ and $u_A = v_A$. Furthermore, if $u_A = v_A$, thus

$L_A u_A = L_A^* u_A = \lambda_1(A) u_A$ and hence $\mathbf{V} \cdot \nabla u_A = 0$, which further implies that

$$\begin{cases} -\operatorname{div}(a(x)\nabla u_A) + c(x)u_A = \lambda_1(A)u_A & \text{in } \Omega, \\ u_A > 0 & \text{in } \Omega, \\ bu_A + (1-b)[a(x)\nabla u_A] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence, $\lambda_1(A) = \lambda_1(0)$. The Claim is proved.

Before proceeding further to show $\frac{\partial \lambda_1}{\partial A}(A) > 0$ for all $A > 0$, let us calculate $\frac{\partial^2 \lambda_1}{\partial A^2}(0)$ firstly. Differentiate equation (9) with respect to A again, and applying the notation $\frac{\partial^2 u_A}{\partial A^2} = u_A''$ for brevity arrives at

$$\begin{cases} -\operatorname{div}[a(x)\nabla u_A''] + A\mathbf{V} \cdot \nabla u_A'' + 2\mathbf{V} \cdot \nabla u_A' + c(x)u_A'' \\ = \frac{\partial^2 \lambda_1}{\partial A^2}(A)u_A + 2\frac{\partial \lambda_1}{\partial A}(A)u_A' + \lambda_1(A)u_A'' & \text{in } \Omega, \\ bu_A'' + (1-b)[a(x)\nabla u_A''] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (11)$$

Setting $A = 0$ in (11) and multiplying it by u_0 and integrating the result in Ω , it follows from $\frac{\partial \lambda_1}{\partial A}(0) = 0$ that

$$\frac{\partial^2 \lambda_1}{\partial A^2}(0) = 2 \int_{\Omega} u_0 \mathbf{V} \cdot \nabla u_0' dx.$$

On the other hand, multiplying equation (9) by u_0' and setting $A = 0$, we have

$$\begin{aligned} & \frac{b}{1-b} \int_{\partial\Omega} (u_0')^2 dS_x + \int_{\Omega} \nabla u_0' \cdot [a(x)\nabla u_0'] dx - \int_{\Omega} u_0 \mathbf{V} \cdot \nabla u_0' dx + \int_{\Omega} c(x)(u_0')^2 dx \\ & = \lambda_1(0) \int_{\Omega} (u_0')^2 dx, \end{aligned}$$

which in turn implies that

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 \lambda_1}{\partial A^2}(0) &= \frac{b}{1-b} \int_{\partial\Omega} (u_0')^2 dS_x + \int_{\Omega} \nabla u_0' \cdot [a(x)\nabla u_0'] dx + \int_{\Omega} c(x)(u_0')^2 dx \\ &\quad - \lambda_1(0) \int_{\Omega} (u_0')^2 dx. \end{aligned} \quad (12)$$

We are now in a position to prove Theorem 1.1. According to the above Claim, it suffices to prove that $\lambda_1(A) > \lambda_1(0)$ for every $A > 0$. If $\lambda_1(\hat{A}) = \lambda_1(0)$ for some $\hat{A} > 0$, since $\frac{\partial \lambda_1}{\partial A}(A) \geq 0$, $\lambda_1(A) \equiv \lambda_1(0)$ for $A \in [0, \hat{A}]$. Thus $\frac{\partial^2 \lambda_1}{\partial A^2}(0) = 0$. By (12) we have

$$\lambda_1(0) = \frac{\frac{b}{1-b} \int_{\partial\Omega} (u_0')^2 dS_x + \int_{\Omega} \nabla u_0' \cdot [a(x)\nabla u_0'] dx + \int_{\Omega} c(x)(u_0')^2 dx}{\int_{\Omega} (u_0')^2 dx},$$

so the variational argument of principal eigenvalue $\lambda_1(0)$ implies that $u_0' = cu_0$ for some constant c . Setting $A = 0$ and then substituting equality $u_0' = cu_0$ into equation (9), we can conclude that $\mathbf{V} \cdot \nabla u_0 \equiv 0$ in Ω , which is a contradiction. This completes the proof. $\square$

We now proceed to prove Theorem 1.2.

Proof of Theorem 1.2. It suffices to establish the following result:

Claim. Assume that $\mathcal{I}_b \neq \emptyset$. Then $\lambda_1(A)$ is uniformly bounded and

$$\lambda_1(A) \leq \inf_{\omega \in \mathcal{I}_b} \frac{\frac{b}{1-b} \int_{\partial\Omega} \omega^2 dS_x + \int_{\Omega} \nabla \omega \cdot [a(x)\nabla \omega] dx + \int_{\Omega} c(x)\omega^2 dx}{\int_{\Omega} \omega^2 dx}, \quad \forall A \geq 0.$$

The idea of the proof for the Claim comes from Theorem 2.2 in [5] and we shall sketch the proof for the sake of completeness. Note that $u_A > 0$ in $\bar{\Omega}$ by Hopf Boundary Lemma for case of $0 \leq b < 1$. Choose any function $\omega \in \mathcal{I}_b$ and multiply the equation of u_A by $\frac{\omega^2}{u_A}$, then integration by parts implies that

$$\begin{aligned} & \frac{b}{1-b} \int_{\partial\Omega} \omega^2 dS_x + \int_{\Omega} \nabla \left(\frac{\omega^2}{u_A} \right) \cdot [a(x) \nabla u_A] dx + A \int_{\Omega} \omega^2 \mathbf{V} \cdot \nabla \log u_A dx \\ & + \int_{\Omega} c \omega^2 dx = \lambda_1(A) \int_{\Omega} \omega^2 dx. \end{aligned} \quad (13)$$

An interesting observation, in analogy with the proof of Theorem 2.2 in [5], gives

$$\int_{\Omega} \omega^2 \mathbf{V} \cdot \nabla \log u_A dx = 0 \text{ and } \int_{\Omega} \nabla \left(\frac{\omega^2}{u_A} \right) \cdot [a(x) \nabla u_A] dx \leq \int_{\Omega} \nabla \omega \cdot [a(x) \nabla \omega] dx,$$

which leads to the Claim by combining equality (13) and $\mathcal{I}_b \neq \emptyset$.

It turns out that $\mathcal{I}_b \neq \emptyset$ always holds for $0 \leq b < 1$, since it at least follows that $c \in \mathcal{I}_b$ for any constant c . Together with the above Claim, the monotonicity of $\lambda_1(A)$ in Theorem 1.1 readily implies that the limit $\lim_{A \rightarrow \infty} \lambda_1(A)$ always exists and is finite. The proof of Theorem 1.2 is complete. $\square$

Remark 1. (Necessity of the assumption $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$): We now remark that the additional assumption $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$ is necessary for $0 \leq b < 1$, while not necessary for $b = 1$, corresponding to zero Dirichlet boundary condition.

- For $b = 1$, zero Dirichlet boundary condition implies $u_A = v_A = 0$ on $\partial\Omega$ and the adjoint operator of L_A can be written as $L_A^* = -\operatorname{div}(a(x) \nabla) - A \mathbf{V} \cdot \nabla + c(x)$ without the additional assumption, whence Theorem 1.1 remains true as the properties of J_A in Section 2 hold without this assumption as stated in subsection 2.2.
- For $0 \leq b < 1$, Theorem 1.1 may fail without the assumption $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Consider the same example as in Remark 2.5 of [5],

$$\begin{cases} -\varphi_A'' + A\varphi_A' + c(x)\varphi_A = \lambda_1(A)\varphi_A, & 0 < x < 1, \\ \varphi_A'(0) = \varphi_A'(1) = 0. \end{cases}$$

Here we consider the special case where $b = 0$ and the incompressible flow $\mathbf{V} = 1$ does not satisfy the assumption $\mathbf{V} \cdot \mathbf{n} = 0$ at $x = 0, 1$. Chen and Lou's result in [8] implies $\lim_{A \rightarrow +\infty} \lambda_1(A) = c(0)$ by treating $\mathbf{V} = -\nabla(-x)$. Assume further that $c'(x) \geq 0$ and $c(x) \not\equiv \text{constant}$. If Theorem 1.1 holds, since $\lambda_1(0) \geq \min_{x \in [0,1]} c(x) = c(0)$, we have $\lambda_1(A) \equiv c(0)$, and thus $\varphi_0' = 0$ according to part (ii) in Theorem 1.1, which contradicts to $c(x) \not\equiv \text{constant}$.

4. Min-Max characterization of $\lambda_1(A)$. In this section we focus on a new min-max characterization of $\lambda_1(A)$ for elliptic operator L_A with incompressible flow and general boundary conditions. To state our main result, some preparations are needed. In this connection, in view of the classical min-max characterization of principal eigenvalue [38]:

$$\lambda_1(A) = \sup_{\omega \in \mathbb{S}_b} \inf_{x \in \Omega} \left[\frac{L_A \omega(x)}{\omega(x)} \right] = \inf_{\omega \in \mathbb{S}_b} \sup_{x \in \Omega} \left[\frac{L_A \omega(x)}{\omega(x)} \right]$$

together with the facts

$$\inf_{p \in L^2(\Omega), \int_{\Omega} p^2 = 1} \int_{\Omega} p^2(x) \left(\frac{L_A \omega}{\omega} \right) dx = \inf_{x \in \Omega} \left[\frac{L_A \omega(x)}{\omega(x)} \right],$$

and

$$\sup_{p \in L^2(\Omega), \int_{\Omega} p^2 = 1} \int_{\Omega} p^2(x) \left(\frac{L_A \omega}{\omega} \right) dx = \sup_{x \in \Omega} \left[\frac{L_A \omega(x)}{\omega(x)} \right],$$

it is straightforward to derive the following min-max characterization of $\lambda_1(A)$:

$$\begin{aligned} \lambda_1(A) &= \sup_{\omega \in \mathbb{S}_b} \inf_{p \in L^2(\Omega), \int_{\Omega} p^2 = 1} \int_{\Omega} p^2(x) \left(\frac{L_A \omega}{\omega} \right) dx \\ &= \inf_{\omega \in \mathbb{S}_b} \sup_{p \in L^2(\Omega), \int_{\Omega} p^2 = 1} \int_{\Omega} p^2(x) \left(\frac{L_A \omega}{\omega} \right) dx. \end{aligned} \quad (14)$$

However, the min-max characterization in Theorem 1.3 is somewhat different. The following result is the key of the proof of Theorem 1.3:

Lemma 4.1.

$$\sup_{\omega \in \mathbb{S}_b} J_A(\omega) = J_A(u_A) = \lambda_1(A).$$

Furthermore, if $J_A(\omega_0) = \sup_{\omega \in \mathbb{S}_b} J_A(\omega)$ for some $\omega_0 \in \mathbb{S}_b$, then $\omega_0 = cu_A$ for some constant $c > 0$.

Lemma 4.1 is a direct consequence of Proposition 2.2 by recalling the positive definiteness of $a(x)$. With the help of Lemma 4.1, Theorem 1.3 can be proved in a straightforward manner.

Proof of Theorem 1.3. We first choose $p^2 = u_A v_A$ and apply Lemma 4.1 to deduce

$$\lambda_1(A) = \sup_{\omega \in \mathbb{S}_b} \int_{\Omega} u_A v_A \left(\frac{L_A \omega}{\omega} \right) dx \geq \inf_{p \in L^2(\Omega), \int_{\Omega} p^2 = 1} \sup_{\omega \in \mathbb{S}_b} \int_{\Omega} p^2(x) \left(\frac{L_A \omega}{\omega} \right) dx.$$

On the other hand, for any $p \in L^2(\Omega)$ satisfying $\int_{\Omega} p^2 = 1$, it is easy to see that

$$\lambda_1(A) = \int_{\Omega} p^2(x) \left(\frac{L_A u_A}{u_A} \right) dx \leq \sup_{\omega \in \mathbb{S}_b} \int_{\Omega} p^2(x) \left(\frac{L_A \omega}{\omega} \right) dx,$$

which implies that

$$\lambda_1(A) \leq \inf_{p \in L^2(\Omega), \int_{\Omega} p^2 = 1} \sup_{\omega \in \mathbb{S}_b} \int_{\Omega} p^2(x) \left(\frac{L_A \omega}{\omega} \right) dx.$$

Hence equality (6) holds. The proof of Theorem 1.3 is now complete. $\square$

Remark 2. (Reduce to the classical Rayleigh-Ritz formula) The classical Rayleigh-Ritz formula is actually implicitly contained in the min-max formula in Theorem 1.3 if L_A is self-adjoint, i.e., $\mathbf{V} = 0$ or $A = 0$. It can be deduced from an important result in [14]. More specifically, viewing $\mu = p^2(x)dx$ in (6) as a positive measure satisfying the mild assumption $\mu \ll \lambda$ for the Borel measure λ and noting that $\frac{d\mu}{d\lambda} = p^2(x)$, Theorem 5 in [14] leads to

$$\sup_{\omega \in \mathbb{S}_b} \int_{\Omega} p^2(x) \left(\frac{L_A \omega}{\omega} \right) dx = \int_{\Omega} p(x) L_A p(x) dx,$$

which reduces the formula in Theorem 1.3 to the classical Rayleigh-Ritz formula.

5. Functional J_A on unbounded periodic domain. We now turn to refine functional J_A defined in (2) for unbounded periodic domain by identifying the definition cone $\mathbb{S}$. Different from $\mathbb{S}_b$ in Section 2 for bounded domain, some modifications are required. To this end, we first present general periodic setting considered in [4, 6, 40].

5.1. The general periodic setting. We consider equation (1) in the form of

$$\begin{cases} w_t = \operatorname{div}(a(z)\nabla w) + A\mathbf{V} \cdot \nabla w + f(z, w) & \text{in } \Omega \times (0, \infty), \\ [a(z)\nabla w] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases}$$

Let $N \geq 1$ be the space dimension and d be an integer satisfying $1 \leq d \leq N$. Write $x = (x_1, \dots, x_d)$ and $y = (y_{d+1}, \dots, y_N)$ so that for all $z = (x, y) \in \Omega$, $|y| \leq R$ for some $R \geq 0$. Set $\Omega = \Omega_X \times \Omega_Y$ with some bounded domain $\Omega_Y \subset \mathbb{R}^{N-d}$. And Ω is further assumed to be periodic with respect to x variables in the following sense: there exists $\mathbf{L} = (L_1, L_2, \dots, L_d)$ such that

$$\forall (k_1, k_2, \dots, k_d) \in L_1\mathbb{Z} \times L_2\mathbb{Z} \times \dots \times L_d\mathbb{Z}, \quad \Omega = \Omega + \sum_{i=1}^d k_i e_i,$$

where $\{e_i\}_{1 \leq i \leq N}$ is the canonical basis of $\mathbb{R}^N$. Denote the periodicity cell of Ω by

$$\mathcal{C} = \left\{ (x, y) \in \Omega : x \in \prod_{i=1}^d (0, L_i) \right\}.$$

This framework includes several types of simple geometrical configurations: the whole space $\mathbb{R}^N$ corresponding to $d = N$ in [4, 43], the infinite cylinders to $d = 1$ in [3, 23], and the infinite slabs to $1 < d < N$.

For the coefficients of equation (1) with $z = (x, y)$, we assume that the diffusion matrix a is symmetric and uniformly elliptic $C^{1,\alpha}(\bar{\Omega})$ matrix field satisfying

$$\begin{cases} a(x, y) \text{ is } \mathbf{L}\text{-periodic with respect to } x; \\ \exists 0 < \gamma_1 < \gamma_2, \text{ such that } \gamma_1 |\xi|^2 \leq \xi^T a(x, y) \xi \leq \gamma_2 |\xi|^2, \quad \forall (x, y) \in \Omega, \forall \xi \in \mathbb{R}^N. \end{cases}$$

Also, the vector field $\mathbf{V} \in C^1(\bar{\Omega})$ is assumed to satisfy

$$\begin{cases} \mathbf{V} \text{ is } \mathbf{L}\text{-periodic with respect to } x; \\ \operatorname{div} \mathbf{V} = 0 & \text{in } \bar{\Omega}; \\ \mathbf{V} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega; \\ \forall 1 \leq i \leq d, \quad \int_{\mathcal{C}} \mathbf{V}_i = 0, \end{cases}$$

which means the underlying flow $\mathbf{V}$ is tangent to $\partial\Omega$ and the first d components have been normalized as in [3, 4, 6, 22, 40]. Finally, the reaction $wf(z, w)$ in (1) is the KPP nonlinearity satisfying the conventional condition described, for example, in (1.5) of [6] or (2.4) of [40].

Under the above framework, of particular interest is the minimal speed of the traveling fronts. For the case of $A \neq 0$, i.e., in the presence of incompressible flow $\mathbf{V}$, Berestycki et al. [4, 6] established a variational formula for the minimal speed $c^*(A)$ in any direction $\mathbf{e} \in \mathbb{R}^d$:

$$c^*(A) = \min_{\lambda > 0} \frac{-\mu_1(A, \lambda)}{\lambda}, \quad (15)$$

where $\mu_1(A, \lambda)$ is the principal eigenvalue of the operator

$$\begin{aligned} \mathcal{L}_{A, \lambda} \omega = & -\operatorname{div}[a \nabla \omega] + \lambda \{ \operatorname{div}[a \bar{\mathbf{e}} \omega] + \bar{\mathbf{e}} \cdot [a \nabla \omega] \} + A \mathbf{V} \cdot \nabla \omega \\ & + [-\lambda^2 \bar{\mathbf{e}} \cdot [a \bar{\mathbf{e}}] - A \lambda \mathbf{V} \cdot \bar{\mathbf{e}} + c] \omega \end{aligned}$$

acting on the set

$$E_\lambda = \{\omega \in C^2(\bar{\Omega}) : \omega \text{ is } \mathbf{L}\text{-periodic in } x \text{ and } a[\nabla\omega - \lambda\bar{\mathbf{e}}\omega] \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

with $c = -f(z, 0)$ and $\bar{\mathbf{e}} = (\mathbf{e}^T, \mathbf{0})^T \in \mathbb{R}^N$. The existence of $\mu_1(A, \lambda)$ is guaranteed by the Krein-Rutman Theorem with corresponding positive eigenfunction ω_A . It was shown in Proposition 4.2 of [35] that the minimum in (15) can be attained uniquely at some $\lambda = \lambda^*(A) \in (0, +\infty)$ for any given $A > 0$ such that

$$c^*(A) = \frac{-\mu_1(A, \lambda^*(A))}{\lambda^*(A)}. \quad (16)$$

To proceed further, set $u_A = e^{-\lambda\mathbf{e} \cdot x} \omega_A$. Then $u_A > 0$ satisfies

$$\begin{cases} L_A u_A := -\operatorname{div}[a\nabla u_A] + \mathbf{A}\mathbf{V} \cdot \nabla u_A + c u_A = \mu_1(A, \lambda) u_A & \text{in } \Omega, \\ [a\nabla u_A] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ u_A(0, y) = e^{\lambda\mathbf{e} \cdot \mathbf{L}} u_A(\mathbf{L}, y) \text{ and } \nabla u_A(0, y) = e^{\lambda\mathbf{e} \cdot \mathbf{L}} \nabla u_A(\mathbf{L}, y) & \text{in } \Omega_Y. \end{cases} \quad (17)$$

Similarly, let v_A be a positive eigenfunction associated with the eigenvalue $\mu_1(A, \lambda)$ of the adjoint problem of (17), namely,

$$\begin{cases} L_A^* v_A := -\operatorname{div}[a\nabla v_A] - \mathbf{A}\mathbf{V} \cdot \nabla v_A + c v_A = \mu_1(A, \lambda) v_A & \text{in } \Omega, \\ [a\nabla v_A] \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ v_A(\mathbf{0}, y) = e^{-\lambda\mathbf{e} \cdot \mathbf{L}} v_A(\mathbf{L}, y) \text{ and } \nabla v_A(\mathbf{0}, y) = e^{-\lambda\mathbf{e} \cdot \mathbf{L}} \nabla v_A(\mathbf{L}, y) & \text{in } \Omega_Y. \end{cases} \quad (18)$$

We further normalize u_A and v_A without loss of generality such that $\int_{\mathcal{C}} u_A^2 = \int_{\mathcal{C}} u_A v_A = 1$.

5.2. Functional J_A on the periodic setting. As a crucial step in the proof of Theorem 1.4, we first introduce the functional on the periodic setting in this section. Define set $\hat{\mathbb{S}} = \{\varphi \in C^1(\bar{\Omega}) : [a\nabla\varphi] \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$ with $\mathbb{S}^0 = \{\varphi \in \hat{\mathbb{S}} : \varphi > 0 \text{ in } \Omega\}$, and further write

$$\begin{cases} \mathbb{S}_{per+}^\lambda = \{\varphi \in \hat{\mathbb{S}} : e^{\lambda\mathbf{e} \cdot x} \varphi \text{ is } \mathbf{L}\text{-periodic with respect to } x\}, \\ \mathbb{S}_{per-}^\lambda = \{\varphi \in \hat{\mathbb{S}} : e^{-\lambda\mathbf{e} \cdot x} \varphi \text{ is } \mathbf{L}\text{-periodic with respect to } x\}. \end{cases}$$

We then identify $\mathbb{S} = \mathbb{S}^0 \cap (\mathbb{S}_{per+}^\lambda \cup \mathbb{S}_{per-}^\lambda)$, i.e., define functional J_A on periodicity cell $\mathcal{C}$ by

$$J_A(\omega) = \int_{\mathcal{C}} u_A v_A \left(\frac{L_A \omega}{\omega} \right), \quad \omega \in \mathbb{S}^0 \cap (\mathbb{S}_{per+}^\lambda \cup \mathbb{S}_{per-}^\lambda). \quad (19)$$

We show a crucial property of J_A , which is different from Proposition 2.2.

Proposition 5.1. *For all $\omega \in \mathbb{S}_{per-}^\lambda$, the following formula holds:*

$$J_A(u_A) - J_A(\omega) = \int_{\mathcal{C}} u_A v_A \left\{ \left[\nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \left[a \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} - 2\lambda \frac{\partial \mu_1}{\partial \lambda}. \quad (20)$$

Proof. As in (7), we begin with rewriting functional J_A as

$$\begin{aligned}
 J_A(\omega) &= - \int_{\mathcal{C}} u_A v_A \left[\frac{\operatorname{div}(a \nabla \omega)}{\omega} \right] + A \int_{\mathcal{C}} u_A v_A \left[\frac{\mathbf{V} \cdot \nabla \omega}{\omega} \right] + \int_{\mathcal{C}} u_A v_A c \\
 &= - \sum_{i=1}^d \int_{\Omega_Y} \{ u_A v_A (a \nabla \log \omega)_i \} \Big|_{(\mathbf{0}, y)}^{(\mathbf{L}, y)} dy + \int_{\mathcal{C}} \nabla \left(\frac{u_A v_A}{\omega} \right) \cdot [a \nabla \omega] \\
 &\quad + A \int_{\mathcal{C}} u_A v_A \mathbf{V} \cdot \nabla \log \omega + \int_{\mathcal{C}} u_A v_A c \\
 &= - \int_{\mathcal{C}} u_A v_A \left\{ (\nabla \log \omega) \cdot [a \nabla \log \omega] \right\} \\
 &\quad + \int_{\mathcal{C}} [A u_A v_A \mathbf{V} + a \nabla (u_A v_A)] \cdot \nabla \log \omega + \int_{\mathcal{C}} u_A v_A c.
 \end{aligned} \tag{21}$$

Here we used the assumption $\omega \in \mathbb{S}_{per+}^\lambda \cup \mathbb{S}_{per-}^\lambda$ to remove boundary integrals by noting the periodicity of $u_A v_A (a \nabla \log \omega)$.

By (21), we may write the Fréchet derivation $J'_A(u_A)$ in the form of

$$\begin{aligned}
 &J'_A(u_A) \varphi \\
 &= - 2 \int_{\mathcal{C}} u_A v_A \left\{ (\nabla \log u_A) \cdot \left[a \nabla \left(\frac{\varphi}{u_A} \right) \right] \right\} \\
 &\quad + \int_{\mathcal{C}} [A u_A v_A \mathbf{V} + a \nabla (u_A v_A)] \cdot \nabla \left(\frac{\varphi}{u_A} \right),
 \end{aligned} \tag{22}$$

for all $\varphi \in \mathbb{S}_{per+}^\lambda$. Following the same arguments in Lemma 2.1, we can conclude from the definition (19) of J_A that

$$J'_A(u_A) \varphi = 0 \text{ for all } \varphi \in \mathbb{S}_{per+}^\lambda. \tag{23}$$

To obtain formula (20), using (21) and by direct calculation we have

$$\begin{aligned}
 &J_A(u_A) - J_A(\omega) \\
 &= \int_{\mathcal{C}} u_A v_A \left\{ (\nabla \log \omega) \cdot [a \nabla \log \omega] \right\} - \int_{\mathcal{C}} u_A v_A \left\{ (\nabla \log u_A) \cdot [a \nabla \log u_A] \right\} \\
 &\quad - \int_{\mathcal{C}} [A u_A v_A \mathbf{V} + a \nabla (u_A v_A)] \cdot \nabla \log \left(\frac{\omega}{u_A} \right) \\
 &= \int_{\mathcal{C}} u_A v_A \left\{ \left[\nabla \log (u_A \omega) \right] \cdot \left[a \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} \\
 &\quad - \int_{\mathcal{C}} [A u_A v_A \mathbf{V} + a \nabla (u_A v_A)] \cdot \nabla \log \left(\frac{\omega}{u_A} \right) \\
 &= \int_{\mathcal{C}} u_A v_A \left\{ \left[\nabla \log \left(\frac{\omega}{u_A} \right) + 2 \nabla \log u_A \right] \cdot \left[a \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} \\
 &\quad - \int_{\mathcal{C}} [A u_A v_A \mathbf{V} + a \nabla (u_A v_A)] \cdot \nabla \log \left(\frac{\omega}{u_A} \right) \\
 &= \int_{\mathcal{C}} u_A v_A \left\{ \left[\nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \left[a \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} \\
 &\quad + 2 \int_{\mathcal{C}} u_A v_A \left\{ (\nabla \log u_A) \cdot \left[a \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\}
 \end{aligned}$$

$$- \int_{\mathcal{C}} \left[Au_A v_A \mathbf{V} + a \nabla(u_A v_A) \right] \cdot \nabla \log \left(\frac{\omega}{u_A} \right),$$

where we have used $v_A \in \mathbb{S}_{per-}^\lambda$ and $u_A \in \mathbb{S}_{per+}^\lambda$, and the symmetry of matrix field a . We now turn to verify formula (20). Choose

$$\varphi = u_A \left[\log \left(\frac{\omega}{u_A} \right) - 2\lambda \mathbf{e} \cdot x \right]$$

in (22) because of $u_A \left[\log \left(\frac{\omega}{u_A} \right) - 2\lambda \mathbf{e} \cdot x \right] \in \mathbb{S}_{per+}^\lambda$ for $\omega \in \mathbb{S}_{per-}^\lambda$. We further compute

$$\begin{aligned} & J_A(u_A) - J_A(\omega) \\ &= \int_{\mathcal{C}} u_A v_A \left\{ \left[\nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \left[a \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} - J'_A(u_A) \varphi + 2\lambda \int_{\mathcal{C}} Au_A v_A \mathbf{V} \cdot \bar{\mathbf{e}} \\ & \quad + 2\lambda \int_{\mathcal{C}} \left\{ v_A [a \nabla u_A] \cdot \bar{\mathbf{e}} - u_A [a \nabla v_A] \cdot \bar{\mathbf{e}} \right\} \\ &= \int_{\mathcal{C}} u_A v_A \left\{ \left[\nabla \log \left(\frac{\omega}{u_A} \right) \right] \cdot \left[a \nabla \log \left(\frac{\omega}{u_A} \right) \right] \right\} + 2\lambda G, \end{aligned}$$

where we used (23) to remove $J'_A(u_A) \varphi$, and G is given by

$$G = \int_{\mathcal{C}} Au_A v_A \mathbf{V} \cdot \bar{\mathbf{e}} + \int_{\mathcal{C}} \left\{ v_A [a \nabla u_A] \cdot \bar{\mathbf{e}} - u_A [a \nabla v_A] \cdot \bar{\mathbf{e}} \right\}.$$

To assert formula (20), it is now desirable to show

Claim. $G = -\frac{\partial \mu_1}{\partial \lambda}$.

To establish our assertion, multiplying (17) by $v_A \mathbf{e} \cdot x$, we derive by virtue of $\mathbf{V} \cdot \mathbf{n} = 0$ on $\partial\Omega$ that

$$\begin{aligned} & -\mathbf{e} \cdot \mathbf{L} \sum_{i=1}^d \int_{\Omega_Y} \left\{ v_A (a \nabla u_A)_i \right\} dy \Big|_{x=\mathbf{L}} + \int_{\mathcal{C}} \mathbf{e} \cdot x \left\{ \nabla v_A \cdot [a \nabla u_A] \right\} \\ & + \int_{\mathcal{C}} \left\{ v_A [a \nabla u_A] \cdot \bar{\mathbf{e}} \right\} - \int_{\mathcal{C}} Au_A v_A \mathbf{V} \cdot \bar{\mathbf{e}} \\ &= \int_{\mathcal{C}} \mathbf{e} \cdot x \left\{ u_A [A \mathbf{V} \cdot \nabla v_A - c v_A + \mu_1 v_A] \right\} - \mathbf{e} \cdot \mathbf{L} A \sum_{i=1}^d \int_{\Omega_Y} \left\{ u_A v_A V_i \right\} dy \Big|_{x=\mathbf{L}}. \end{aligned}$$

Again integrating by parts and inferring from the definition (18) of v_A yields that

$$\begin{aligned} & -\mathbf{e} \cdot \mathbf{L} \sum_{i=1}^d \int_{\Omega_Y} \left\{ v_A (a \nabla u_A)_i \right\} dy \Big|_{x=\mathbf{L}} + \mathbf{e} \cdot \mathbf{L} \sum_{i=1}^d \int_{\Omega_Y} \left\{ u_A (a \nabla v_A)_i \right\} dy \Big|_{x=\mathbf{L}} \\ & + \underbrace{\int_{\mathcal{C}} \left\{ v_A [a \nabla u_A] \cdot \bar{\mathbf{e}} \right\} - \int_{\mathcal{C}} Au_A v_A \mathbf{V} \cdot \bar{\mathbf{e}} - \int_{\mathcal{C}} \left\{ u_A [a \nabla v_A] \cdot \bar{\mathbf{e}} \right\}}_{-G} \\ &= -\mathbf{e} \cdot \mathbf{L} A \sum_{i=1}^d \int_{\Omega_Y} \left\{ u_A v_A V_i \right\} dy \Big|_{x=\mathbf{L}}, \end{aligned}$$

which leads directly to

$$\begin{aligned} G = & -\mathbf{e} \cdot \mathbf{L} \sum_{i=1}^d \int_{\Omega_Y} \left\{ v_A (a \nabla u_A)_i - u_A (a \nabla v_A)_i \right\} dy \Big|_{x=\mathbf{L}} \\ & + \mathbf{e} \cdot \mathbf{L} A \sum_{i=1}^d \int_{\Omega_Y} \left\{ u_A v_A V_i \right\} dy \Big|_{x=\mathbf{L}}. \end{aligned} \quad (24)$$

To proceed further, we now turn to calculate $\frac{\partial \mu_1}{\partial \lambda}$. Differentiate (17) with respect to λ and denote $\frac{\partial u_A}{\partial \lambda} = u'_A$ for the sake of brevity, we obtain

$$\begin{cases} -\operatorname{div} [a \nabla u'_A] + A \mathbf{V} \cdot \nabla u'_A + c u'_A = \frac{\partial \mu_1}{\partial \lambda} u_A + \mu_1 u'_A & \text{in } \Omega, \\ [a \nabla u'_A] \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \text{ and } \int_C u'_A u_A = 0, \\ u'_A(\mathbf{0}, y) = \mathbf{e} \cdot \mathbf{L} e^{\lambda \mathbf{e} \cdot \mathbf{L}} u_A(\mathbf{L}, y) + e^{\lambda \mathbf{e} \cdot \mathbf{L}} u'_A(\mathbf{L}, y) & \text{in } \Omega_Y, \\ \nabla u'_A(\mathbf{0}, y) = \mathbf{e} \cdot \mathbf{L} e^{\lambda \mathbf{e} \cdot \mathbf{L}} \nabla u_A(\mathbf{L}, y) + e^{\lambda \mathbf{e} \cdot \mathbf{L}} \nabla u'_A(\mathbf{L}, y) & \text{in } \Omega_Y. \end{cases} \quad (25)$$

We may multiply this equation by v_A to derive that

$$\begin{aligned} & - \sum_{i=1}^d \int_{\Omega_Y} \left\{ v_A (a \nabla u'_A)_i \right\} \Big|_{(\mathbf{0}, y)}^{(\mathbf{L}, y)} dy + \sum_{i=1}^d \int_{\Omega_Y} \left\{ u'_A (a \nabla v_A)_i \right\} \Big|_{(\mathbf{0}, y)}^{(\mathbf{L}, y)} dy \\ & + A \sum_{i=1}^d \int_{\Omega_Y} \left\{ v_A u'_A V_i \right\} \Big|_{(\mathbf{0}, y)}^{(\mathbf{L}, y)} dy \\ & = \frac{\partial \mu_1}{\partial \lambda} + \mu_1 \int_C v_A u'_A - \int_C \{ -\operatorname{div} [a \nabla v_A] - A \mathbf{V} \cdot \nabla v_A + c v_A \} u'_A. \end{aligned} \quad (26)$$

Together with the definition (18) of v_A and the boundary condition of u'_A in (25), the Claim follows from (24) and (26). Formula (20) is therefore verified. $\square$

6. Monotonicity of $c^*(A)/A$. Our goal of this section is to prove Theorem 1.4. Before proceeding further, we require some necessary properties of principal eigenvalue $\mu_1(A, \lambda)$ to verify the differentiability of $c^*(A)$.

6.1. Concavity of $\mu_1(A, \lambda)$ with respect to λ . This subsection is devoted to describing some properties related to the convexity of $\mu_1(A, \lambda)$ as a function of λ . The concavity of the map $\lambda \mapsto \mu_1(A, \lambda)$ has been already proved in [4], but we focus here on the strict concavity. We further prove $\frac{\partial^2 \mu_1}{\partial \lambda^2}(A, \lambda) < 0$, which appears to be new and it is a key to derive the differentiability of $c^*(A)$ with respect to A .

Lemma 6.1. *The principal eigenvalue $\mu_1(A, \lambda)$ is strictly concave in $\lambda > 0$.*

Proof. As already noted, the proof of the concavity of $\mu_1(A, \lambda)$ has been carried out by the general lines of [4]. Here we just outline it for completeness and because it will lead us to the strict concavity.

Set $\lambda_1, \lambda_2 \in \mathbb{R}$, $t \in (0, 1)$ and $\lambda = t\lambda_1 + (1-t)\lambda_2$. Denote by $u^{\lambda_1} \in \mathbb{S}^0 \cap \mathbb{S}_{per+}^{\lambda_1}$, $u^{\lambda_2} \in \mathbb{S}^0 \cap \mathbb{S}_{per+}^{\lambda_2}$ the principal eigenfunctions associated with λ_1 and λ_2 , respectively. We further define $\varphi^{\lambda_1} = \log(u^{\lambda_1})$, $\varphi^{\lambda_2} = \log(u^{\lambda_2})$, $\varphi^\lambda = t\varphi^{\lambda_1} + (1-t)\varphi^{\lambda_2}$, and $u^\lambda = e^{\varphi^\lambda} \in \mathbb{S}^0 \cap \mathbb{S}_{per+}^\lambda$ accordingly. Direct calculation yields that

$$\frac{L_A u^\lambda}{u^\lambda} = -\operatorname{div} [a \nabla \varphi^\lambda] - \nabla \varphi^\lambda \cdot [a \nabla \varphi^\lambda] + A \mathbf{V} \cdot \nabla \varphi^\lambda + c$$

$$\begin{aligned}
&\geq t \left\{ -\operatorname{div} [a \nabla \varphi^{\lambda_1}] - \nabla \varphi^{\lambda_1} \cdot [a \nabla \varphi^{\lambda_1}] + A \mathbf{V} \cdot \nabla \varphi^{\lambda_1} + c \right\} \\
&\quad + (1-t) \left\{ -\operatorname{div} [a \nabla \varphi^{\lambda_2}] - \nabla \varphi^{\lambda_2} \cdot [a \nabla \varphi^{\lambda_2}] + A \mathbf{V} \cdot \nabla \varphi^{\lambda_2} + c \right\} \\
&= t \left[\frac{L_A u^{\lambda_1}}{u^{\lambda_1}} \right] + (1-t) \left[\frac{L_A u^{\lambda_2}}{u^{\lambda_2}} \right],
\end{aligned} \tag{27}$$

and the equality holds if and only if $\nabla \varphi^{\lambda_1} = \nabla \varphi^{\lambda_2}$ because of the positivity and symmetry of matrix field a . By (27), the concavity of $\mu_1(A, \lambda)$ follows readily from the max-min formula for principal eigenvalue of elliptic operators [37],

$$\mu_1(A, \lambda) = \max_{u^\lambda \in \mathbb{S}^0 \cap \mathbb{S}_{per+}^\lambda} \min_c \left[\frac{L_A u^\lambda}{u^\lambda} \right].$$

We then turn to consider the strict concavity of $\mu_1(A, \lambda)$ in λ . Suppose the equality in (27) holds. Then $\nabla \varphi^{\lambda_1} = \nabla \varphi^{\lambda_2}$ implies that $\varphi^{\lambda_1} = \varphi^{\lambda_2} + c$, and thus $u^{\lambda_1} = cu^{\lambda_2}$ for some $c > 0$. In view of $u^{\lambda_1} \in \mathbb{S}_{per+}^{\lambda_1}$ and $u^{\lambda_2} \in \mathbb{S}_{per+}^{\lambda_2}$, we can find $e^{\lambda_1 \mathbf{e} \cdot \mathbf{L}} u^{\lambda_1}(\mathbf{L}, y) = u^{\lambda_1}(\mathbf{0}, y) = cu^{\lambda_2}(\mathbf{0}, y) = ce^{\lambda_2 \mathbf{e} \cdot \mathbf{L}} u^{\lambda_2}(\mathbf{L}, y) = e^{\lambda_2 \mathbf{e} \cdot \mathbf{L}} u^{\lambda_1}(\mathbf{L}, y)$, $\forall y \in \Omega_Y$, which asserts $\lambda_1 = \lambda_2$. Accordingly the strict concavity has been proved. $\square$

We are now in a position to establish a stronger result that is the main objective of this subsection.

Proposition 6.2. *For all $A > 0, \lambda > 0$, $\frac{\partial^2 \mu_1}{\partial \lambda^2}(A, \lambda) < 0$ holds.*

Proof. Proposition 6.2 will be derived as a byproduct of Lemma 6.1, along with the maximum principle. By taking $\varphi_A = \log u_A$, the eigenvalue problem (17) becomes

$$\begin{cases} -\operatorname{div}[a \nabla \varphi_A] + A \mathbf{V} \cdot \nabla \varphi_A - \nabla \varphi_A \cdot [a \nabla \varphi_A] + c = \mu_1(A, \lambda) & \text{in } \Omega, \\ [a \varphi_A] \cdot \mathbf{n} = 0 & \text{on } \partial \Omega, \\ \varphi_A(\mathbf{0}, y) = \lambda \mathbf{e} \cdot \mathbf{L} + \varphi_A(\mathbf{L}, y) \text{ and } \nabla \varphi_A(\mathbf{0}, y) = \nabla \varphi_A(\mathbf{L}, y) & \text{in } \Omega_Y. \end{cases}$$

We then take two derivatives with respect to λ and $\varphi_A'' := \frac{\partial^2 \varphi_A}{\partial \lambda^2}$ satisfies

$$\begin{cases} -\operatorname{div}[a \nabla \varphi_A''] + (A \mathbf{V} - 2a \nabla \varphi_A) \cdot \nabla \varphi_A'' - 2 \nabla \varphi_A' \cdot [a \nabla \varphi_A'] = \frac{\partial^2 \mu_1}{\partial \lambda^2}(A, \lambda) & \text{in } \Omega, \\ [a \varphi_A''] \cdot \mathbf{n} = 0 & \text{on } \partial \Omega, \\ \varphi_A''(\mathbf{0}, y) = \varphi_A''(\mathbf{L}, y) \text{ and } \nabla \varphi_A''(\mathbf{0}, y) = \nabla \varphi_A''(\mathbf{L}, y) & \text{in } \Omega_Y. \end{cases} \tag{28}$$

Lemma 6.1 asserts that $\frac{\partial^2 \mu_1}{\partial \lambda^2}(A, \lambda) \leq 0$. We now prove the desired result by contradiction argument and assume that there is some $\lambda_0 > 0$ such that $\frac{\partial^2 \mu_1}{\partial \lambda^2}(A, \lambda_0) = 0$. Restricting the equation (28) at λ_0 , we find that

$$-\operatorname{div}[a \nabla \varphi_A''] + (A \mathbf{V} - 2a \nabla \varphi_A) \cdot \nabla \varphi_A'' \geq 0 \quad \text{in } \Omega.$$

Denote (x_0, y_0) by $\varphi_A''(x_0, y_0) = \min_{\bar{\mathcal{C}}} \varphi_A''$. The strong maximum principle asserts that $(x_0, y_0) \in \partial \mathcal{C}$. Recalling the boundary conditions of φ_A'' in (28), the Hopf boundary lemma implies that φ_A'' is a constant and thus $\nabla \varphi_A'' = 0$. Returning to (28), $\nabla \varphi_A'' = 0$ reads $\nabla \varphi_A' = 0$ at the point λ_0 .

Proceeding to differentiate (28) in λ gives

$$\begin{cases} -\operatorname{div}[a \nabla \varphi_A'''] + (A \mathbf{V} - 2a \nabla \varphi_A) \cdot \nabla \varphi_A''' - 6 \nabla \varphi_A' \cdot [a \nabla \varphi_A'] = \frac{\partial^3 \mu_1}{\partial \lambda^3}(A, \lambda) & \text{in } \Omega, \\ [a \varphi_A'''] \cdot \mathbf{n} = 0 & \text{on } \partial \Omega, \\ \varphi_A'''(\mathbf{0}, y) = \varphi_A'''(\mathbf{L}, y) \text{ and } \nabla \varphi_A'''(\mathbf{0}, y) = \nabla \varphi_A'''(\mathbf{L}, y) & \text{in } \Omega_Y. \end{cases}$$

Again setting $\lambda = \lambda_0$ and regarding $\nabla \varphi'_A = \nabla \varphi''_A = 0$ at λ_0 , we arrive at

$$-\operatorname{div}[a\nabla \varphi'''_A] + (A\mathbf{V} - 2a\nabla \varphi_A) \cdot \nabla \varphi'''_A = \frac{\partial^3 \mu_1}{\partial \lambda^3}(A, \lambda_0) \quad \text{in } \Omega.$$

This actually implies $\nabla \varphi'''_A = 0$ and thus $\frac{\partial^3 \mu_1}{\partial \lambda^3}(A, \lambda_0) = 0$. To prove this assertion, there is no loss of generality in assuming $\frac{\partial^3 \mu_1}{\partial \lambda^3}(A, \lambda_0) \geq 0$. The boundary condition of φ'''_A then asserts that φ'''_A is a constant by the strong maximum principle and Hopf boundary lemma again.

Repeating the above procedure, we may conclude that

$$\nabla \varphi_A^{(n)} = 0 \quad \text{and} \quad \frac{\partial^n \mu_1}{\partial \lambda^n}(A, \lambda_0) = 0 \quad \text{for all } n \geq 2.$$

Notice the well-known fact that $\mu_1(A, \lambda)$ is analytic. See, e.g. Proposition 2.20 in [7]. The Taylor expansion of $\mu_1(A, \lambda)$ at λ_0 reads

$$\mu_1(A, \lambda) = \mu_1(A, \lambda_0) + \frac{\partial \mu_1}{\partial \lambda}(A, \lambda_0)(\lambda - \lambda_0),$$

which contradicts to the strict concavity in Lemma 6.1. $\square$

6.2. Proof of Theorem 1.4. With the help of Propositions 5.1 and 6.2, we are in a position to prove Theorem 1.4.

Proof. Differentiating (17) with respect to A and writing $\frac{\partial u_A}{\partial A} = u'_A$ to obtain

$$\begin{cases} -\operatorname{div}[a\nabla u'_A] + A\mathbf{V} \cdot \nabla u'_A + \mathbf{V} \cdot \nabla u_A + cu'_A = \frac{\partial \mu_1}{\partial A}(A, \lambda)u_A + \mu_1(A, \lambda)u'_A & \text{in } \Omega, \\ [a\nabla u'_A] \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \text{ and } \int_{\mathcal{C}} u'_A u_A = 0, \\ u'_A(\mathbf{0}, y) = e^{\lambda \mathbf{e} \cdot \mathbf{L}} u'_A(\mathbf{L}, y) \text{ and } \nabla u'_A(\mathbf{0}, y) = e^{\lambda \mathbf{e} \cdot \mathbf{L}} \nabla u'_A(\mathbf{L}, y) & \text{in } \Omega_Y. \end{cases}$$

Multiplying both sides of the above equation by v_A and integrating by parts gives

$$\frac{\partial \mu_1}{\partial A}(A, \lambda) = \int_{\mathcal{C}} v_A \mathbf{V} \cdot \nabla u_A.$$

Recalling the definitions of L_A , L_A^* , and functional J_A , we further derive

$$\begin{aligned} \frac{\partial \mu_1}{\partial A}(A, \lambda) &= \frac{1}{2A} \int_{\mathcal{C}} v_A (L_A - L_A^*) u_A \\ &= \frac{1}{2A} \left[\int_{\mathcal{C}} v_A L_A u_A - \int_{\mathcal{C}} u_A L_A v_A \right] \\ &= \frac{1}{2A} [J_A(u_A) - J_A(v_A)] \\ &= \frac{1}{2A} \int_{\mathcal{C}} u_A v_A \left\{ \left[\nabla \log \left(\frac{v_A}{u_A} \right) \right] \cdot \left[a \nabla \log \left(\frac{v_A}{u_A} \right) \right] \right\} - \frac{\lambda}{A} \frac{\partial \mu_1}{\partial \lambda}(A, \lambda), \end{aligned} \tag{29}$$

by appealing to Proposition 5.1 with $\omega = v_A \in \mathbb{S}_{per-}^\lambda$ particularly. As already noted in Subsection 5.1, there exists a unique point $\lambda^*(A) \in (0, +\infty)$ satisfying (16), whence the map $A \mapsto \lambda^*(A)$ is well defined. At $\lambda^*(A)$, by the argument of extreme point we have

$$\lambda^*(A) \frac{\partial \mu_1}{\partial \lambda}(A, \lambda^*(A)) = \mu_1(A, \lambda^*(A)), \tag{30}$$

which establishes, along with (16), the following

$$\frac{\partial \mu_1}{\partial A}(A, \lambda^*(A)) = \frac{1}{2A} \int_{\mathcal{C}} u_A v_A \left\{ \left[\nabla \log \left(\frac{v_A}{u_A} \right) \right] \cdot \left[a \nabla \log \left(\frac{v_A}{u_A} \right) \right] \right\} - \frac{1}{A} \mu_1(A, \lambda^*(A)). \quad (31)$$

On the other hand, consider function $F(A, \lambda) = \lambda \frac{\partial \mu_1}{\partial \lambda}(A, \lambda) - \mu_1(A, \lambda)$. Since $F(A, \lambda^*(A)) = 0$ and $\frac{\partial F(A, \lambda)}{\partial \lambda} = \frac{\partial^2 \mu_1}{\partial \lambda^2}(A, \lambda) \neq 0$ by Proposition 6.2, the implicit function theorem implies $\lambda^*(A)$, and hence $c^*(A)$, are differentiable with respect to A . In view of equalities (16), (30), and (31), direct calculation leads to

$$\begin{aligned} \frac{dc^*(A)}{dA} &= \frac{-\frac{\partial \mu_1}{\partial A}(\lambda^*(A))}{\lambda^*(A)} \\ &= -\frac{1}{2A\lambda^*(A)} \int_{\mathcal{C}} u_A v_A \left\{ \left[\nabla \log \left(\frac{v_A}{u_A} \right) \right] \cdot \left[a \nabla \log \left(\frac{v_A}{u_A} \right) \right] \right\} + \frac{c^*(A)}{A}. \end{aligned}$$

This yields the desired result

$$\frac{d(c^*(A)/A)}{dA} = -\frac{1}{2A^2\lambda^*(A)} \int_{\mathcal{C}} u_A v_A \left\{ \left[\nabla \log \left(\frac{v_A}{u_A} \right) \right] \cdot \left[a \nabla \log \left(\frac{v_A}{u_A} \right) \right] \right\}. \quad (32)$$

To complete the proof of Theorem 1.4, it suffices to show that $\frac{d(c^*(A)/A)}{dA} < 0$. Suppose not, there exists $A_0 > 0$ so that $u_{A_0} = cv_{A_0}$ for some $c > 0$. The normalization that $\int_{\mathcal{C}} u_A^2 = 1$ and $\int_{\mathcal{C}} u_A v_A = 1$ imply readily that $u_A = v_A$. Then one can apply the periodicity conditions of u_A and v_A to find $\lambda^*(A_0) = 0$, which is a contradiction. Therefore, the function $c^*(A)/A$ is strictly decreasing as $A > 0$ increases, which proves Theorem 1.4. $\square$

7. Discussions and open questions. A similar result to Theorem 1.1 was established in our work [28], which is associated with the periodic-parabolic eigenvalue problem

$$\begin{cases} \tau u_t - \operatorname{div} [a(x) \nabla u] + c(x, t)u = \lambda_1(\tau)u, & x \in \Omega, t \in [0, 1], \\ bu + (1 - b) [a(x) \nabla u] \cdot \mathbf{n} = 0, & x \in \partial\Omega, t \in [0, 1], \\ u(x, 0) = u(x, 1), & x \in \Omega. \end{cases} \quad (33)$$

Such problem remains an important area of active research [24, 25, 31, 33, 35, 36], with particular interest on the dependence of the principal eigenvalue $\lambda_1(\tau)$ on frequency τ . It is shown in [28] that $\lambda_1(\tau)$ is non-decreasing in $\tau > 0$, and more precisely,

- (i) If $c(x, t) = \int_0^1 c(x, s) ds + g(t)$ for some 1-periodic function $g(t)$, then $\lambda_1(\tau)$ is constant for $\tau > 0$;
- (ii) Otherwise $\frac{\partial \lambda_1}{\partial \tau}(\tau) > 0$ for every $\tau > 0$.

Identifying the connections between problems (3) and (33) is an interesting subject.

We now consider problems (3) with gradient flow $\mathbf{V}_1 = \nabla m$ for some $m \in C^2(\bar{\Omega})$, where the principal eigenvalue $\lambda_1(A)$, in analogy with equality (1.2) in [5], can be written as

$$\lambda_1(A) = \inf_{\omega \in H^1(\Omega) \setminus \{0\}} \frac{\frac{b}{1-b} \int_{\partial\Omega} \omega^2 dS_x + \int_{\Omega} \nabla \omega \cdot [a(x) \nabla \omega] + \int_{\Omega} \left(\frac{A^2}{4} |\mathbf{V}_1|^2 - \frac{A}{2} \operatorname{div} \mathbf{V}_1 + c(x) \right) \omega^2}{\int_{\Omega} \omega^2 dx},$$

which implies the monotonicity of $\lambda_1(A)$ if $\mathbf{V}_1$ is incompressible satisfying $\operatorname{div} \mathbf{V}_1 = 0$. This result can be covered by Theorem 1.1 with the extra assumption $\mathbf{V}_1 \cdot \mathbf{n} = 0$ on $\partial\Omega$. However, if the gradient flow $\mathbf{V}_1 = \nabla m$ is incompressible and satisfies $\mathbf{V}_1 \cdot \mathbf{n} = 0$ on $\partial\Omega$, the only possibility is $m = \text{constant}$. Hence we may ask naturally: When does the monotonicity property remain true for gradient flow? Understanding the monotonicity of $\lambda_1(A)$ with general flows seems to be more difficult.

Another open question is to determine the limit value of $\lambda_1(A)$ for incompressible flow $\mathbf{V}$ with Robin boundary conditions as $A \rightarrow +\infty$, though the existence of the limit has been shown in Theorem 1.2. The results for Dirichlet and Neumann boundary conditions in [5] show that the limit of $\lambda_1(A)$ can be determined by the variational principle (4). In view of Theorem 1.2, it seems plausible to conjecture that for $0 \leq b < 1$,

$$\lim_{A \rightarrow +\infty} \lambda_1(A) = \inf_{\omega \in \mathcal{L}_b} \frac{\frac{b}{1-b} \int_{\partial\Omega} \omega^2 dS_x + \int_{\Omega} \nabla \omega \cdot [a(x) \nabla \omega] dx + \int_{\Omega} c(x) \omega^2 dx}{\int_{\Omega} \omega^2 dx},$$

which would reduce to the results in [5] for the case $b = 0$. On the other hand, the limit value of $\lambda_1(A)$ with the gradient flow $\mathbf{V}_1 = \nabla m$ has been established by Chen and Lou [8] for Neumann boundary conditions, which can be stated as

$$\lim_{A \rightarrow +\infty} \lambda_1(A) = \min_{\mathcal{M}} c,$$

with the set $\mathcal{M}$ consisting of all points of local maximum of m . Hence a natural question arises: Does the limit of $\lambda_1(A)$ exist as $A \rightarrow +\infty$ for general flows under proper boundary conditions? If it exists, what is the limit value?

There are a substantial body of literatures concerning the asymptotic behavior of the principal eigenvalue of elliptic operators for small diffusion rates; See [9, 11, 12, 16, 42]. For the principal eigenvalue of operator $L_D = -D\Delta + \mathbf{V} \cdot \nabla + c(x)$, Chen and Lou [9] investigated its asymptotic behavior as $D \rightarrow 0$ when $\mathbf{V}$ is a gradient flow. Much less seems to be known when $\mathbf{V}$ is a general incompressible flow; See [2, 41].

Acknowledgments. We thank the referee for helpful suggestions. We thank Professor Nadin for bringing Reference [19] to our attention. We thank Professors Berestycki and Hamel for their helpful discussions.

REFERENCES

- [1] I. Averill, K. Y. Lam and Y. Lou, [The role of advection in a two-species competition model: A bifurcation approach](#), *Mem. Amer. Math. Soc.*, **245** (2017), v+117 pp.
- [2] J. Bedrossian and M. C. Zelati, [Enhanced dissipation, hypoellipticity, and anomalous small noise inviscid limits in shear flows](#), *Arch. Rational Mech. Anal.*, **224** (2017), 1161–1204.
- [3] H. Berestycki, [The influence of advection on the propagation of fronts in reaction-diffusion equations](#), *Nonlinear PDE's in Condensed Matter and Reactive Flows*, **569** (2002), 11–48.
- [4] H. Berestycki and F. Hamel, [Front propagation in periodic excitable Media](#), *Comm. Pure Appl. Math.*, **55** (2002), 949–1032.
- [5] H. Berestycki, F. Hamel and N. Nadirashvili, [Elliptic eigenvalue problems with large drift and applications to nonlinear propagation phenomena](#), *Comm. Math. Phys.*, **253** (2005), 451–480.
- [6] H. Berestycki, F. Hamel and N. Nadirashvili, [The speed of propagation for KPP-type problems. I. Periodic framework](#), *J. Eur. Math. Soc.*, **7** (2005), 173–213.
- [7] R. S. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, Wiley Series in Mathematical and Computational Biology, John Wiley and Sons, Chichester, 2003.
- [8] X. F. Chen and Y. Lou, [Principal eigenvalue and eigenfunctions of an elliptic operator with large advection and its application to a competition model](#), *Indiana Univ. Math. J.*, **57** (2008), 627–658.

- [9] X. F. Chen and Y. Lou, [Effects of diffusion and advection on the smallest eigenvalue of an elliptic operators and their applications](#), *Indiana Univ. Math. J.*, **61** (2012), 45–80.
- [10] P. Constantin, A. Kiselev, L. Ryzhik and A. Zlatoš, [Diffusion and mixing in fluid flow](#), *Ann. Math.*, **168** (2008), 643–674.
- [11] A. Devinatz, R. Ellis and A. Friedman, [The asymptotic behavior of the first real eigenvalue of second order elliptic operators with a small parameter in the highest derivatives, II](#), *Indiana Univ. Math. J.*, **23** (1973/74), 991–1011.
- [12] A. Devinatz and A. Friedman, [Asymptotic behavior of the principal eigenfunction for a singularly perturbed Dirichlet problem](#), *Indiana Univ. Math. J.*, **27** (1978), 143–157.
- [13] M. D. Donsker and S. R. S. Varadhan, [On a variational formula for the principal eigenvalue for operators with maximum principle](#), *Proc. Natl. Acad. Sci. U.S.A.*, **72** (1975), 780–783.
- [14] M. D. Donsker and S. R. S. Varadhan, [Asymptotic evaluation of certain Markov process expectations for large time, I](#), *Comm. Pure Appl. Math.*, **28** (1975), 1–47.
- [15] M. D. Donsker and S. R. S. Varadhan, [On the principal eigenvalue of second-order elliptic differential operators](#), *Comm. Pure Appl. Math.*, **29** (1976), 595–621.
- [16] A. Fannjiang, [The asymptotic behavior of the first real eigenvalue of a second order elliptic operator with a small parameter in the highest derivatives](#), *Indiana Univ. Math. J.*, **22** (1973), 1005–1015.
- [17] A. Fannjiang and G. Papanicolaou, [Convection enhanced diffusion for periodic flows](#), *SIAM J. Appl. Math.*, **54** (1994), 333–408.
- [18] T. Godoy, J. P. Gossez and S. Paczka, [A minimax formula for the principal eigenvalues of Dirichlet problems and its applications](#), Proceedings of the 2006 International Conference in honor of Jacqueline Fleckinger, *Electron. J. Differ. Equ. Conf., Texas State Univ.–San Marcos, Dept. Math., San Marcos, TX*, **16** (2007), 137–154.
- [19] T. Godoy, J. P. Gossez and S. Paczka, [On the asymptotic behavior of the principal eigenvalues of some elliptic problems](#), *Ann. Mat. Pur. Appl.*, **189** (2010), 497–521.
- [20] F. Hamel, [Qualitative properties of monostable pulsating fronts: Exponential decay and monotonicity](#), *J. Math. Pures Appl.*, **89** (2008), 355–399.
- [21] F. Hamel and N. Nadirashvili, [Extinction versus persistence in strong oscillating flows](#), *Arch. Rational Mech. Anal.*, **195** (2010), 205–223.
- [22] F. Hamel and A. Zlatoš, [Speed-up of combustion fronts in shear flows](#), *Math. Ann.*, **356** (2013), 845–867.
- [23] S. Heinze, [Large convection limits for KPP fronts](#), Max Planck Institute for Mathematics Preprint, 2005.
- [24] V. Hutson, K. Michaikow and P. Poláčik, [The evolution of dispersal rates in a heterogeneous time-periodic environment](#), *J. Math. Biol.*, **43** (2001), 501–533.
- [25] V. Hutson, W. Shen and G.T. Vickers, [Estimates for the principal spectrum point for certain time-dependent parabolic operators](#), *Proc. Amer. Math. Soc.*, **129** (2001), 1669–1679.
- [26] G. Iyer, A. Novikov, L. Ryzhik and A. Zlatoš, [Exit times of diffusions with incompressible drift](#), *SIAM J. Math. Anal.*, **42** (2009), 2484–2498.
- [27] A. Kiselev, R. Shterenberg and A. Zlatoš, [Relaxation enhancement by time-periodic flows](#), *Indiana Univ. Math. J.*, **57** (2008), 2137–2152.
- [28] S. Liu, Y. Lou, R. Peng and M. Zhou, [Monotonicity of the principal eigenvalue for a linear time-periodic parabolic operator](#), *Proc. Amer. Math. Soc.*, 2019.
- [29] T. Ma and S. Wang, [Structural classification and stability of divergence-free vector fields](#), *Physics D*, **171** (2002), 107–126.
- [30] P. Michel, S. Mischler and B. Perthame, [General relative entropy inequality: An illustration on growth models](#), *J. Math. Pures Appl.*, **84** (2005), 1235–1260.
- [31] G. Nadin, [The principal eigenvalue of a space-time periodic parabolic operator](#), *Ann. Mat. Pura Appl.*, **188** (2009), 269–295.
- [32] G. Nadin, [The effect of the Schwarz rearrangement on the periodic principal eigenvalue of a nonsymmetric operator](#), *SIAM J. Math. Anal.*, **41** (2009/10), 2388–2406.
- [33] G. Nadin, [Some dependence results between the spreading speed and the coefficients of the space-time periodic Fisher-KPP equation](#), *Euro. J. Appl. Math.*, **22** (2011), 169–185.
- [34] W.-M. Ni, [The Mathematics of Diffusion](#), CBMS-NSF Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

- [35] J. Nolen, M. Rudd and J. Xin, [Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds](#), *Dyn. Partial Differ. Equ.*, **2** (2005), 1–24.
- [36] J. Nolen and J. Xin, [Reaction-diffusion front speeds in spatially-temporally periodic shear flows](#), *Multiscale Model. Simul.*, **1** (2003), 554–570.
- [37] R. D. Nussbaum and Y. Pinchover, [On variational principles for the generalized principal eigenvalue of second order elliptic operators and some applications](#), Festschrift on the occasion of the 70th birthday of Shmuel Agmon, *J. Anal. Math.*, **59** (1992), 161–177.
- [38] M. H. Protter and H. F. Weinberger, [On the spectrum of general second order operators](#), *Bull. Amer. Math. Soc.*, **72** (1966), 251–255.
- [39] M. Sion, [On general minimax theorems](#), *Pacific J. Math.*, **8** (1958), 171–176.
- [40] M. E. Smailly and S. Kirsch, [Front speed enhancement by incompressible flows in three or higher dimensions](#), *Arch. Rational Mech. Anal.*, **213** (2014), 327–354.
- [41] J. Vukadinovic, E. Dedits, A. C. Poje and T. Schäfer, [Averaging and spectral properties for the 2D advection-diffusion equation in the semi-classical limit for vanishing diffusivity](#), *Physics D*, **310** (2015), 1–18.
- [42] A. D. Wentzell, [On the asymptotic behavior of the first eigenvalue of a second order differential operator with small parameter in higher derivatives](#), *Theory Prob. Appl.*, **20** (1975), 599–602.
- [43] J. X. Xin, [Existence of planar flame fronts in convective-diffusive periodic media](#), *Arch. Ration. Mech. Anal.*, **121** (1992), 205–233.
- [44] A. Zlatoš, [Sharp asymptotics for KPP pulsating front speed-up and diffusion enhancement by flows](#), *Arch. Rational Mech. Anal.*, **195** (2010), 441–453.

Received September 2018; revised April 2019.

E-mail address: liushuangnqkg@ruc.edu.cn

E-mail address: lou@math.ohio-state.edu