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Article · August 2002

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The influence of advection on the propagation of fronts in reaction-diffusion equations

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July 10, 2002

1 Introduction

Propagation of fronts is a phenomenon which plays a central role in a varied array of different fields. Front solutions in combustion represent propagating flames in particular in the setting of deflagrations in premixed gases (see e.g. [13, 68]). In physics and chemistry, more generally, propagating fronts describe phase transitions as a steady transformation taking place at a well defined velocity. Biological invasions or changes in populations are also often modelled as fronts (see e.g. P. Fife [26], J. Murray [53] and Shigesada and Kawasaki [62]). Propagation of fronts and of pulses appears indeed to be a very general phenomenon in *excitable media*.

In many situations, the pertinent mathematical models describing these phenomena are *reaction–diffusion* type equations or systems. Here, I shall only discuss situations which are represented by a scalar function $u = u(t, x)$ and the general *homogeneous* reaction–diffusion equation is

$$u_t - \Delta u = f(u) \quad \text{in} \quad \mathbb{R}^N \quad (1.1)$$

where the term Δu describes diffusion (or random dispersion) and $f(u)$ is the term representing the reaction (related to the excitability of the medium).

When there is an underlying flow $\vec{q}(x)$ which gives rise to a transport of the scalar u , one is led to the reaction–diffusion–advection equation:

$$u_t - \Delta u + \vec{q}(x) \cdot \vec{\nabla} u = f(u) \quad (1.2)$$

The homogeneous equation (1.1) above was introduced in the works of Fisher [28] in 1937 to describe spreading phenomena in population genetics with the

logistics law for the reaction term $f(u) = u(1 - u)$. Almost immediately, this equation was the object of the fundamental article of Kolmogorov, Petrovsky and Piskunov [46] which laid the ground for the study of nonlinear parabolic equations and introduced some of the essential tools in this field. Simultaneously, in 1939, Zeldovich and Frank-Kamenetski (see [76, 77]) proposed this type of equation (with a different type of reaction term f) as the simplest model to study flame propagation in the framework of the *thermo-diffusive* approximation in combustion.

Ever since these works, this type of equations (and systems) has been introduced in a fascinating array of different fields, in particular in the context of biology and ecology.

The original work of Zeldovich and Frank-Kamenetski as well as several subsequent ones is about equation (1.1). In combustion however, the hydrodynamical aspects are of foremost importance and it is essential to deal with equation (1.2) taking $\vec{q}$ as given as a first step to understand the complete system which couples the thermodynamics, chemical kinetics and fluid mechanics aspects of the problem, where instead of advection, one deals with convection.

Here, I will focus on recent developments dealing with the advection reaction-diffusion equations (1.2). To understand the influence of the advection term $\vec{q}$ on the propagation of fronts plays indeed a centerpiece role in combustion. Of particular interest is the influence of large advection terms on the shape of fronts and the speed at which they propagate. In combustion this is a first step to understand some aspects of turbulent combustion.

Note that in contradistinction with (1.1), equation (1.2) is spatially heterogeneous in general. Likewise, if instead of considering the equation (1.1) in all of space, one considers propagation in a domain $\Omega \subset \mathbb{R}^N$, that is :

$$u_t - \Delta u = f(u) \quad \text{in } \Omega \tag{1.3}$$

the problem that one is led to in general is not homogeneous any more. This heterogeneous aspect will be important here. To address this issue, I shall consider here the framework of propagation of *periodic media*, that is when the term $\vec{q}$ in (1.2) or the domain Ω (or both) have periodic structures. More general equations involving periodic diffusion coefficients or a nonlinear reaction term $f = f(x, u)$ which has a periodic dependence in x will be mentioned later on.

Regarding another approach to heterogeneous media, many results have also been obtained in the framework of *random media*. These will not be covered here and the reader is referred to the book of Freidlin [30] and to the survey of J. Xin [?] for this aspect as well as many other related questions for periodic media.

I shall first recall in the next section some classical results dealing with the homogeneous equations and planar fronts. A particular emphasis will be put on the properties regarding long time behavior and asymptotic spreading for these equations. In this connection, I shall present in section 3 a simplified proof of a symmetrization result due to C. Jones.

Higher dimensional problems already arise for curved fronts, when there is a parallel (shear) flow $\vec{q}$ in (1.2). In this case, considered in section 4, one can still define traveling fronts, which are given by some nonlinear elliptic problem. In section 5, I shall present some new results (which have not appeared elsewhere) regarding the speed of traveling fronts in shear flows with large advection. In the KPP case (to be defined below), one obtains a complete description of the dependence of the critical speed in terms of the amplitude of the flow. In the following section, some properties related to the dynamics of the problem will be recalled.

In the general periodic setting, traveling fronts do not exist any more and one is led to extend this notion. I shall define in section 7 the *pulsating traveling fronts* also called periodic traveling fronts. A large part of the theory of planar fronts can be extended to this generalized setting. The existence results, presented in section 8 are taken from joint work with François Hamel [9]. The question of the speed of propagation of fronts through a periodic array of rapid vortical cells is an important question in turbulent combustion. I shall present some results and open problems regarding this question in section 9. Lastly, in section 10, I shall give various estimates regarding the speed of propagation, emphasizing in particular the influence of geometry and advection. Along this survey, I shall indicate several open problems.

This paper is intended as a survey of some of these developments. It is the written form of a series of lectures given at the NATO Advanced Scientific Institute in Cargese in the summer of 1999. Some of this material was also presented in a course at the Pacific Institute of Mathematical Sciences (UBC) in Vancouver in the summer 2001. I have also included some recent results on the subject that have appeared since.

In preparing the Cargese lectures and these notes, I much benefited from

several discussions with Paul Clavin, François Hamel, Louis Nirenberg, Yves Pomeau and Jean-Michel Roquejoffre. It is a pleasure to express my thankfulness to them.

2 Planar fronts and large time behavior for homogeneous reaction–diffusion equations

I first review some results about the homogeneous reaction–diffusion equation (1.1). This will be the occasion to recall the classical theory of planar fronts and their role in the large time dynamics for these equations. There are three different aspects here :

- Existence of traveling fronts
- Convergence to traveling fronts,
- Asymptotic spreading from initial data with compact support.

The second topic is mainly restricted to one spatial dimension. The point there is to consider the initial value problem:

$$\begin{aligned} u_t - u_{xx} &= f(u), & x \in \mathbb{R} \\ u(0, x) &= u_0(x) \end{aligned} \tag{2.1}$$

and to determine the convergence of solutions of (2.1) to traveling fronts. Some indications will also be given regarding higher dimensions. For the asymptotic speed of spreading, the equation under consideration is (1.1) in arbitrary dimension. Planar fronts play an essential role in understanding the asymptotic spreading.

I shall recall the results of existence of planar fronts together with the long time behavior in (2.1). Asymptotic spreading will be considered separately.

2.1 Planar fronts

Throughout this article, it will be assumed that $f : [0, 1] \rightarrow \mathbb{R}$ is a Lipschitz continuous function which is of class C^1 in some neighbourhoods of 0 and 1 and satisfies $f(0) = f(1) = 0$. Fronts are solutions which connect the two stationary states $u \equiv 0$ and $u \equiv 1$.

Planar traveling fronts propagating in a direction $\vec{e}$ (where $\vec{e}$ is a unit vector) are such solutions of (1.1) of the form

$$u(t, x) = \phi(x \cdot \vec{e} + ct). \quad (2.2)$$

Then, clearly, ϕ satisfies

$$-\phi'' + c\phi' = f(\phi), \quad \phi(-\infty) = 0, \quad \phi(+\infty) = 1. \quad (2.3)$$

Three types of nonlinearity are usually considered. I indicate the relevant results for each one of them.

2.2 KPP reaction term

First we consider the case of KPP (for Kolmogorov, Petrovski and Piskunov) type nonlinearities. It is defined by assuming that $f > 0$ in $(0, 1)$ and is strictly concave (or more generally satisfies (4.6) below). The example of such a nonlinearity is the one in Fisher's equation, that is $f(u) = u(1 - u)$. In this case, there is a continuum of possible speeds: $[c^* = \sqrt{2f'(0)}, +\infty[$. More precisely, no planar front exists in (2.3) with a speed $c < c^*$, while for each $c \geq c^*$, there exists a unique solution ϕ of (2.3). It is understood here that fronts take their values in the interval $(0, 1)$. Uniqueness in this contexts is always understood up to shifts of the origin.

Since one can always shift the origin in (2.3), denote by ϕ_c the front with speed c and such that $\phi_c(0) = 1/2$.

We start with convergence results for front like initial data. The first result of this kind is the following.

Theorem 2.1 (Kolmogorov, Petrovski, Piskunov, (1937), [46])

Assume u_0 to be the Heaviside function $H(x)$. Then there is a function $x(t)$ with $\lim_{t \rightarrow +\infty} \dot{x}(t) = 0$ such that for all $x \in \mathbb{R}$

$$\lim_{t \rightarrow +\infty} (u(t, x) - \phi_{c_*}(x + c_*t + x(t))) = 0.$$

The proof of this theorem in [?] relies on deep considerations about the number of zeroes of the solutions of a parabolic equation which much later gave rise to several works on this question.

The initial datum in Theorem 2.1 is special, and one may ask what happens if one considers more general conditions at $\pm\infty$. It turns out that the problem sharply depends on what happens at $-\infty$ - which is the *leading edge* of the front. The reason why this happens is related to the fact that 0 is a saddle for (2.3). For all $c > c_*$, the wave ϕ_c decays near $-\infty$ at the rate

$$\phi_c(x) \sim ke^{r_-(c)x}, \quad \text{with} \quad r_-(c) = \frac{c - \sqrt{c^2 - 4f'(0)}}{2}$$

Thus, clearly, Theorem 2.1 cannot hold for general initial data just assuming that $u_0 \rightarrow 1$ as $x \rightarrow \infty$ and $u_0 \rightarrow 0$ as $x \rightarrow -\infty$.

A typical result is the following

Theorem 2.2 (Bramson [19] 1983, Uchiyama [64] 1978, Lau [47] 1985)) *Assume $u_0(+\infty) = 1$, and $u_0(x) \sim e^{r_-(c)x}$ as $x \rightarrow -\infty$. Then $u(t, x)$ converges to a wave of speed c .*

The memoir of Bramson [19] explores the topic by probabilistic methods. In [47], Lau actually gives necessary and sufficient conditions on the initial datum u_0 to ensure the convergence to a wave of speed c . The proofs in [47] are deterministic and rest on a particular version of the zeroes number principle. This principle states that the number of zeroes of a solution to a linear homogeneous parabolic problem is non increasing with time.

When $c = c_*$, let $x(t)$ be the point such that $u(t, x(t)) = \frac{1}{2}$ (assume e.g. that u_0 is nondecreasing). Precise estimates of $x(t)$ have been derived by Uchiyama ([64], 1978) by probabilistic methods showing that $x(t) \sim k \text{Log} t$.

For a partial generalization of these results - i.e. asymptotic speed - to quasilinear equations, see Hagan [36].

This type of estimates has been considerably generalized recently in a series of works by Ebert and Von Saarloos [24]. They obtain in particular further expansions on $x(t)$ and give more general conditions. They also consider other type of equations. It is an open problem to find an analytical proof for these results. In fact even for Uchiyama's result there is still no analytical approach.

The concavity assumption on f can be relaxed. Namely, we still assume $f > 0$ on $[0, 1]$, $f(0) = f(1) = 0$, $f'(0) > 0$ and $f'(1) < 0$. The concavity assumption on f , however, is dropped. This case is referred to as the ZFK

(Zeldovich-Frank-Kamenetskii) case. Regarding the existence of fronts, there is a continuum of possible speeds $[c_*, +\infty[$, just as in the KPP case. The following results are due to Stokes [63]:

- if $c > c_*$ the situation is similar to that of Theorem 2.2.
- If $c = c_*$ and $c_* > \sqrt{2f'(0)}$, then there is exponential convergence to a wave, provided that the initial datum decays faster than the wave.

Lastly, it is also useful to have stability results. An important result in this direction is due to D. Sattinger:

Theorem 2.3 (Stability of traveling fronts with higher speeds: Sattinger [61] 1976))

Choose $c > c_*$ and suppose u_0/ϕ'_c decays exponentially as $|x| \rightarrow +\infty$. Then

$$u(t, x) - \phi_c(x + ct) = O(e^{-\delta t}) \quad \text{for some } \delta > 0.$$

Note that here there is *no* spatial shift. This is in striking contrast with the bistable case described below for which one proves *orbital* stability.

2.3 Combustion nonlinearity.

This type of nonlinearity is defined by assuming that $f(u) = 0$ on $[0, \theta]$ for some $0 < \theta < 1$ and that $f(u) > 0$ on $(\theta, 1)$ with $f(1) = 0$ and $f'(1) < 0$. In such a case there is a unique solution (c, ϕ) to (2.3) - the profile ϕ being unique up to a translation. Normalize the translation by taking ϕ to be the unique front such that $\phi(0) = \theta$. This type of nonlinearity is used classically in flame models. The number θ is to be understood as a threshold value at which the reaction is turned on. It is sometimes called an *ignition temperature*. The analogue of the KPP result (Theorem 2.1) by Kanel.

Theorem 2.4 (Kanel [44], 1965)

Assume $u_0(x) - H(x)$ to have compact support. Then there exists $x_0 \in \mathbb{R}$ such that the following holds uniformly in $x \in \mathbb{R}$:

$$\lim_{t \rightarrow +\infty} (u(t, x) - \phi(x + ct + x_0)) = 0.$$

This result can be extended in two directions: first, it is enough for the initial datum u_0 to only have exponential decay at $-\infty$, and some convergence to 1 at $+\infty$ in a weak sense. Second, Sattinger's [61] orbital stability results are valid on this model, and the convergence to a wave can be proved to be exponential. In fact, the situation is rather similar to the bistable situation below.

2.4 Bistable nonlinearity.

In the bistable case, the nonlinearity is assumed to satisfy the following conditions for some $\theta \in (0, 1)$: $f(u) < 0$ on $(0, \theta)$; $f(u) > 0$ on $(\theta, 1)$; $f(0) = f(1) = 0$ and $f'(0) < 0$, $f'(1) < 0$. Like in the combustion case, there is a unique speed c and a unique traveling front ϕ solutions of (2.3). Let ϕ denote the unique front such that $\phi(0) = \theta$.

Theorem 2.5 (Fife-McLeod [27], 1977) *Assume:*

$$\limsup_{x \rightarrow -\infty} u_0(x) < \theta \quad \text{and} \quad \liminf_{x \rightarrow +\infty} u_0(x) > \theta$$

Then there exists $x_0 \in \mathbb{R}$ such that

$$u(t, x) - \phi(x + ct + x_0) = O(e^{-\delta t})$$

for some $\delta > 0$.

2.5 Spreading results in one space dimension

I now turn to results dealing with asymptotic spreading. Here one considers initial data u_0 with compact support and $u_0 \geq 0$. The question is to know whether and at which speed the corresponding solution of the initial value problem (2.1) (or the higher dimensional version) converges to 1. When it does converges to 1, one says that there is spreading. Also, one says that there is extinction when $u(t, x) \rightarrow 0$ as $t \rightarrow \infty$.

Some general results about spreading in arbitrary dimension are given in the next paragraph. In one space dimension, they were derived in a first article by Aronson-Weinberger [1].

In the case of one space dimension, one can prove more precise convergence results that I mention now.

- In the KPP case, Rothe [59] proved the convergence to two fronts with opposite speeds when u_0 decays to 0 as $x \rightarrow +\pm\infty$. Similar results are also to be found in [64].
- In the bistable case, Fife-McLeod [27] proved the convergence to fronts with opposite speeds when u_0 is compactly supported, the part of u_0 above θ is large enough, and the front connecting 0 on the left to 1 on the right moves with positive speed ($c > 0$).

2.6 Spreading in higher space dimensions

Consider the initial value problem associated with (1.1). The problem reads:

$$\begin{aligned} u_t - \Delta u &= f(u), & x \in \mathbb{R}^N, \\ u(0, x) &= u_0(x). \end{aligned} \quad (2.4)$$

The initial datum u_0 is assumed to have compact support, or, more generally to decay to 0 as $|x| \rightarrow +\infty$. Here I mention several aspects related to the behavior of the solution of (2.4) for large time.

2.6.1 KPP and the “hair trigger effect”

Assume that f is a KPP-type nonlinearity, and $c^* = 2\sqrt{f'(0)}$ is the minimal speed of traveling fronts in this case.

Theorem 2.6 (Aronson-Weinberger [2], 1978)

Assume that u_0 takes values in $[0, 1]$, has compact support and $u_0 \not\equiv 0$. Then, for all unit vector $\vec{e} \in \mathbb{R}^N$, $|\vec{e}| = 1$:

$$\begin{aligned} \forall c < c^*, \quad & \lim_{t \rightarrow +\infty} u(t, x + ct\vec{e}) = 1 \\ \forall c > c^*, \quad & \lim_{t \rightarrow +\infty} u(t, x + ct\vec{e}) = 0 \end{aligned}$$

2.6.2 Bistable case

We make the same assumptions on f as in the one dimensional case above. Denote by c^* the unique velocity of planar fronts. If $\int_0^1 f(u) du > 0$, then $c^* > 0$.

Theorem 2.7 (Aronson-Weinberger, *ibid.*)

Suppose that the initial data u_0 satisfies the same conditions as above, namely that $u_0(x) \in [0, 1]$ for all $x \in \mathbb{R}^N$ and that u_0 has compact support with $u_0 \not\equiv 0$. Assume further that $c^ > 0$.*

1. *Assume $\{u_0 > \theta\}$ is large enough; then spreading occurs. For all $\vec{e} \in \mathbb{R}^N$, $|\vec{e}| = 1$:*

$$\begin{aligned} \forall c < c^*, \quad & \lim_{t \rightarrow +\infty} u(t, x + ct\vec{e}) = 1 \\ \forall c > c^*, \quad & \lim_{t \rightarrow +\infty} u(t, x + ct\vec{e}) = 0 \end{aligned}$$

2. *Under the condition that $\{u_0 > \theta\}$ is sufficiently small, there is extinction. That is, for all $x \in \mathbb{R}^N$,*

$$\lim_{t \rightarrow +\infty} u(t, x) = 0.$$

When spreading occurs as in case 1 above, it is natural to ask what is the shape of the asymptotic profile. There are two important results of C. Jones on this question.

Let us start with the spherically symmetric case.

Theorem 2.8 (Jones [43], 1985)

Assume again that f is of the bistable type. Under the same conditions on u_0 as in Theorem 2.7 above, assume further that u_0 has spherical symmetry and that spreading occurs. There is a planar front ϕ (that is a solution of (2.3) such that, for all $\vec{e} \in \mathbb{R}^N$, $|\vec{e}| = 1$:

$$\lim_{t \rightarrow +\infty} (u(t, x + ct\vec{e}) - \phi(|x| + c^*t)) = 0.$$

In an earlier paper [42], Jones considered the case where u_0 does not have the radial symmetry, and proved asymptotic spherical symmetry: the level lines of u become spherical in the following sense.

Theorem 2.9 (Jones' symmetrization result [42], 1983)

In the bistable case, for a general u_0 with compact support satisfying the conditions of Theorem 2.7 above, assume that spreading occurs. Let a be a fixed number in $(0, 1)$; in particular, a can be arbitrarily close to 1. For any time $t > 0$ and x_0 such that $u(t, x_0) = a$, and $\nabla_x(t, x_0) \neq 0$, the normal line to the level set $\{x; u(t, x) = a\}$ through the point x_0 intersects the support of the initial datum u_0 .

The assumption of spreading implies that for large t , the level set $u(t, \cdot) = a$ moves to infinity. In a rescaled version, this result means that in the limit of $t \rightarrow \infty$, the level sets, once rescaled to be at finite distance, have their normal at every point going through the same fixed point. Therefore, the above result shows an *asymptotic symmetrization* of the solution of the initial value problem (2.4). In a sense, this result is reminiscent, as a parabolic analogue - in a particular case, of the Gidas-Ni-Nirenberg symmetry property for elliptic equations. (See e.g. [17].)

As a consequence of Theorem 2.9, the following property is shown in [42]:

Proposition 2.1 [42] *If $t > 0$ is large enough, then*

$$\lim_{t \rightarrow +\infty} \left| \frac{\nabla u(t, x)}{|\nabla u(t, x)|} - \frac{x}{|x|} \right| = 0$$

as long as the above expression makes sense.

Very little progress has been made since Jones' results on getting a more detailed description of the asymptotic profile. Just recently it was observed by V. Roussier [60] that this result cannot, in general, be extended into a convergence result to a front. That is, the conclusion in Theorem 2.8 does not necessarily hold for initial data u_0 which are not spherically symmetric.

3 Proof of C. Jones' symmetrization result

The proof by C. Jones of Theorem 2.9 above rests on the maximum principle. It borrows the idea of using reflection from the method of moving planes of Alexandrov and Serrin. I present here a simplified version of Jones' proof.

Denote by K the convex hull of the support of u_0 . By assumption, K is compact. Let us argue by contradiction and assume that for some $a \in (0, 1)$, the result does not hold. That is, for some $t_0 > 0$, denoting $S_0 := \{x \in \mathbb{R}^N ; u(t_0, x) = a\}$, we assume that there exists some point $x_0 \in S_0$ such that the normal line N to S_0 through the point x_0 does not meet K . By the separation theorem of convex sets, there exists a hyperplane T containing N that does not meet K . Let H denote the half space containing K and bounded by T .

For all $x \in H$, let $\hat{x}$ be the reflection (mirror image) of x in T . In H , for $t \geq 0$, we define the functions:

$$\begin{aligned} v(t, x) &:= u(t, \hat{x}), \\ w(t, x) &:= u(t, x) - v(t, x). \end{aligned}$$

Then, like u , v is a solution of the equation:

$$v_t - \Delta v = f(v) \quad \text{for } x \in H, \quad t \geq 0.$$

Since f is Lipschitz continuous and $0 \leq u, v \leq 1$, there is some bounded measurable function $c(t, x)$ such that:

$$f(v) - f(u) = c(t, x)(v - u), \quad \forall t \geq 0, \quad \forall x \in H.$$

Hence, w is a solution of a linear equation:

$$w_t - \Delta w - c(t, x)w = 0, \quad \forall t \geq 0, \quad \forall x \in H,$$

with $C \in L^\infty(\mathbb{R}^+ \times H)$. Moreover, by construction:

$$w(t, x) = 0, \quad \forall x \in T, \quad \forall t \geq 0.$$

The assumption that T does not meet K implies that $v(0, x) = 0$, for all $x \in H$ and therefore that:

$$w(0, x) \geq 0 \quad \forall x \in H.$$

From the maximum principle, it follows that:

$$w(t, x) \geq 0, \quad \forall t \geq 0, \quad \forall x \in H.$$

Let ξ denote the normal to H at the point x_0 . Since $w(t_0, x_0) = 0$ and $w \not\equiv 0$, the strong maximum principle shows that:

$$\frac{\partial w}{\partial \xi}(t_0, x_0) = \nabla_x w(t_0, x_0) \cdot \xi < 0.$$

However, ξ being normal to N at x_0 , it is tangent to the level set $S_0 = \{u(t_0, \cdot) = a\}$. Therefore:

$$\frac{\partial w}{\partial \xi}(t_0, x_0) = 0,$$

and Theorem 2.9 is proved by contradiction.

4 Curved traveling fronts for shear flows

I now turn to heterogeneous problems. A particular attention will be devoted here to the role played by advection.

Consider the reaction-diffusion-advection reaction:

$$u_t - \Delta u + \vec{q}(x) \cdot \vec{\nabla} u = f(u) \tag{4.1}$$

in an infinite cylinder $\Sigma = \mathbb{R} \times \omega$ with $\omega \subset \mathbb{R}^{N-1}$. I use the notation $x = (x_1, y)$ for $x \in \Sigma$ with $x_1 \in \mathbb{R}$ and $y = (x_2, \dots, x_N) \in \omega$. Here, ω is assumed to be a smooth bounded domain in $\mathbb{R}^{N-1}$. The outward unit normal derivatives on ω and on Σ are denoted by ν (so that ν is in $\mathbb{R}^{N-1}$ or in $\mathbb{R}^N$ according to the context).

A particular type of flow which is useful to study and for which there still exist traveling fronts is that of *shear flows* or parallel flows. We assume that:

$$\vec{q}(x) = \alpha(y) \vec{e}_1$$

where $\alpha : \bar{\omega} \rightarrow \mathbb{R}$ is some C^1 function. Note that $\vec{q}$ is divergence free. The flow is uniform along the cylinder but not in its cross section. Equation (4.1) thus reads:

$$u_t - \Delta u + \alpha(y) \frac{\partial u}{\partial x_1} = f(u) \quad \text{in } \Sigma \quad (4.2)$$

Assume adiabatic boundary condition, that is:

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Sigma$$

Looking for a Travelling front solution, we set:

$$u(t, x) = \phi(x_1 + ct, y)$$

where $\phi : \bar{\Sigma} \rightarrow \mathbb{R}$. One is thus led to the following *semi-linear elliptic equation in a cylinder*:

$$\begin{cases} -\Delta \phi + (c + \alpha(y)) \frac{\partial \phi}{\partial x_1} = f(\phi) & \text{in } \Sigma, \\ \phi(-\infty, \cdot) = 0, \phi(+\infty, \cdot) = 1, \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \partial \Sigma. \end{cases} \quad (4.3)$$

The limits in (4.3) are to be understood as uniform with respect to $y \in \bar{\omega}$. The unknowns here are the function ϕ and the velocity c . Hence (like the homogeneous problem (2.3)), it is sometimes referred to as a nonlinear eigenvalue problem with c playing the role of an eigenvalue.

The theory of planar fronts has been completely extended to this case and I now present the main results. As in the homogeneous equation, one has to distinguish between several types of functions f . In all the following, we continue to assume that $f : [0, 1] \rightarrow \mathbb{R}$ is a Lipschitz continuous function which is differentiable in some neighborhoods of 0 and 1. In all cases, it is also assumed that:

$$f(0) = f(1) = 0, \quad f'(1) < 0.$$

The results are stated according to the type of nonlinearity.

4.1 Combustion type nonlinearity

Assume that there exists $\theta \in (0, 1)$ such that:

$$f \equiv 0 \quad \text{on } [0, \theta] \quad \text{and} \quad f > 0 \quad \text{on } (\theta, 1). \quad (4.4)$$

This type of nonlinearity is commonly used to represent combustion phenomena. The term f comes from chemical kinetics and is obtained by Arrhenius

law together with the law of mass action. As was mentioned earlier, the assumption about θ is a kind of cut-off or threshold temperature below which the reaction altogether stops. It is often referred to as an *ignition* temperature. It is a convenient assumption in order to avoid the “cold boundary difficulty” (see [15]).

The main result in this case is the following:

Theorem 4.1 [14, 18]. *Given any C^1 function $\alpha(y)$ and under assumption (4.4), there exists a unique velocity c and a unique (up to translation in the x_1 direction) profile ϕ solution of problem (4.3). Moreover, ϕ satisfies $\frac{\partial \phi}{\partial x_1} > 0$ in $\bar{\Sigma}$.*

Existence of a solution was proved with B. Larrouturou and P.L. Lions in [14] (1990). Its uniqueness as well as the monotonicity result are from joint work with L. Nirenberg [18] (1992).

To prove the existence, we considered the problem reduced to a bounded domain $\Sigma_a = (-a, a) \times \omega$, with a normalization condition:

$$\max_{y \in \bar{\omega}} u(0, y) = \theta$$

and boundary conditions on $x_1 = \pm a$. Using Leray-Schauder degree, we obtain a solution of the problem in Σ_a . Then, using *a priori* estimates, we get a solution of (4.3) by taking the limit $a \rightarrow +\infty$. The most delicate point in these estimates is to prove that c_a – the parameter c for the problem in Σ_a – is bounded independently of a :

$$\gamma \leq c_a \leq \Gamma, \quad \forall a \geq a_0.$$

Uniqueness as well as monotonicity are established by using the sliding domain technique of [17] which is similar to the moving plane technique and also rests on the maximum principle.

4.2 Positive and KPP type nonlinearities

The nonlinear term is now assumed to satisfy:

$$f > 0 \quad \text{in } (0, 1) \tag{4.5}$$

In this case too, one can extend the planar front (for (2.3)) theory.

Theorem 4.2 (H. Berestycki, L. Nirenberg [18], 1992) . Assume f satisfies (4.5) and α is a given C^1 function. Then, there exists a critical velocity c^* such that:

- (i) If $c < c^*$, there are no solution of (4.3),
- (ii) For all $c \geq c^*$, there exists a solution ϕ of the corresponding problem (4.3).

If one assumes in addition that $f'(0) > 0$, then, for each $c \geq c^*$, the solution ϕ is unique (up to translations along the axis of the cylinder) and satisfies:

$$\frac{\partial \phi}{\partial x_1} > 0 \quad \text{in } \bar{\Sigma}.$$

In the case $f'(0) = 0$, the questions of uniqueness and of monotonicity are still open for positive nonlinearities.

Let me now define the class of KPP nonlinearities for which a precise result about c^* can be achieved. A nonlinear term f is said to be of KPP type if it satisfies:

$$0 < f(s) \leq f'(0)s, \quad \forall s \in (0, 1) \quad (4.6)$$

For such nonlinearities, one can characterize c^* and this will turn out to be very useful. This is related to the linearization of the problem near $u = 0$ (i.e. as $x_1 \rightarrow -\infty$):

$$-\Delta u + (c + \alpha(y)) \frac{\partial u}{\partial x_1} = f'(0)u$$

If one looks for *exponential solutions* of this equation:

$$u(x_1, y) = e^{\lambda x_1} \varphi(y) \quad x_1 \text{ in } \mathbf{R}, \quad y \in \bar{\omega}$$

with $\lambda > 0$ and $\varphi > 0$ in $\bar{\omega}$, one has to solve the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_y \varphi + \lambda(c + \alpha(y)) \varphi - f'(0) \varphi = \lambda^2 \varphi & \text{in } \omega, \\ \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } \partial \omega. \end{cases} \quad (4.7)$$

It turns out that the solvability of (4.7) completely characterizes c^* as the next result shows.

Theorem 4.3 . Let $f'(0) > 0$ be given. There exists a number $\bar{c}$ such that:

- (i) If $c < \bar{c}$, there are no solution of (4.7) with $\lambda > 0$ and $\varphi > 0$ in ω

- (ii) If $c = \bar{c}$, there is one and only one value of $\lambda > 0$ for which there exists a solution $\varphi > 0$ of (4.7)
- (iii) If $c > \bar{c}$, there are exactly two values of $\lambda > 0$ for which there exist positive “eigenfunctions” φ of (4.7).
- If f satisfies the assumption (4.6), then:

$$c^* = \bar{c}.$$

When $\alpha \equiv 0$, then, clearly, the principal eigenfunction of (4.7) is a constant, *i.e.* $\varphi \equiv 1$ and the eigenvalue problem reduces to:

$$\lambda c - f'(0) = \lambda^2 \tag{4.8}$$

Hence, in this case, the above characterization reads $c^* = 2\sqrt{f'(0)}$. Thus, Theorem 4.3 yields an extension of this speed characterization to nonhomogeneous situations. It is interesting to emphasize that the general property characterizing the critical speed involves an eigenvalue problem. Thus, c^* only depends on $\alpha(y)$, ω and $f'(0)$ but otherwise does *not* depend on the shape of f as long as the assumption (4.6) is satisfied.

Instead of a cylinder $\Sigma = \mathbb{R} \times \omega$, one can consider the equation (4.3) in all of space $\mathbb{R}^n$, assuming $\alpha(y)$ to be periodic. Then, the same results (with the obvious modifications) hold.

Based on probabilistic arguments, Gärtner and Freidlin [32] gave a characterization of the asymptotic speed of spreading (which I shall define in the next section). It can be shown (see [74]) that the two definitions coincide.

If f does not satisfy (4.6), then, the above characterization does not hold any more. Other, characterizations, of min max type have been found, starting with the work of Hadeler and Rothe [33], on one dimension problems, and its extensions by Volpert et al. [65]. Recently higher dimensional versions of these characterizations have been given by Hamel [34] and by Heinze, Papanicolaou and Stevens [40].

In a recent work, Hamel and Nadirashvili [35] have studied the structure of the set of entire solutions of the homogeneous KPP equation in the whole space. They have discovered that the set of solutions is much richer than was earlier thought. In particular, there are many other types of curved fronts for this equation even in the homogeneous case.

4.3 Bistable nonlinearities

The last class of nonlinearities is the bistable case. It is defined by the following requirement: For some $\theta \in (0, 1)$, f satisfies:

$$\begin{cases} f > 0 & \text{in } (\theta, 1) \\ f < 0 & \text{in } (0, \theta) \end{cases} \quad (4.9)$$

It will also be assumed that $\int_0^1 f(s) ds > 0$.

As opposed to the first two cases that we have seen, the extension to higher dimensions of the bistable case involves an additional assumption on the geometry of the domain.

Theorem 4.4 (H. Berestycki, L. Nirenberg [18]). *Under condition (4.9), assume that the cross section of the cylinder ω is convex. Then, there exists a unique c and a unique ϕ (unique up to translations) solution of (4.3). Furthermore ϕ is monotonic in the x_1 -direction.*

The proof does involve the condition of convexity on ω . It makes use of the property that the only stable solutions of Neumann problems in a convex ω are constant. This property is due to Casten and Holland [20]. It turns out that without such an assumption, the above theorem does not necessarily hold.

Theorem 4.5 (H. Berestycki, F. Hamel [8] 2000). *There exists (non convex) domains ω such that under condition (4.9), the problem (4.3) has no solution.*

The counter examples are constructed in [8] by considering hour-glass type domains.

5 Convection by large shear flows in the KPP case

Consider now the problem when the parallel flow has an amplitude factor A , that is,

$$\vec{q}(x) = A\alpha(y) \vec{e}_1$$

where $\alpha : \bar{\omega} \rightarrow \mathbb{R}$ is a fixed C^1 function. It is an important question to know how the amplitude factor affects the traveling front solutions that we

discussed before. I will describe here the asymptotics of fronts speeds for large amplitudes A in the case of a KPP nonlinearity.

It will be assumed throughout this section that $\alpha \not\equiv 0$, and that

$$\int_{\omega} \alpha(y) dy = 0 \quad (5.1)$$

Note that if condition (5.1) is not met, one can always subtract from $\alpha(y)$ a constant (which is added to c) so that it is satisfied.

We assume here that f is of KPP type, that is, f satisfies condition (4.6). Recall that under this assumption, for all $A > 0$, there exists a critical speed, denoted $c^*(A)$ such that:

- For all $c \geq c^*(A)$, there exists a unique (up to translations) traveling front u , solution of (4.3). Moreover, $\frac{\partial u}{\partial x_1} > 0$ in $\bar{\Sigma}$.
- No traveling front solution of (4.3) exists with speed $c < c^*(A)$.

As we shall see in the next section, $c^*(A)$ also represents the asymptotic speed of spreading. Therefore, to understand the influence of large advection on propagation, it is important to describe the behavior of $c^*(A)$ as $A \rightarrow \infty$. This is the content of the next theorem.

Theorem 5.1 . *Suppose $\alpha \not\equiv 0$ satisfies (5.1). Then, as a function of A , $c^*(A)$, the minimal speed of (4.3), satisfies the following properties:*

- (i) $c^*(A)$ is increasing with $A > 0$,
- (ii) $c^*(A)/A$ is decreasing with $A > 0$,
- (iii) $c^*(A)/A$ converges to a positive limit $\rho > 0$ as $A \rightarrow \infty$.

I will now give the proof of this result which is new.

It rests on the characterization of $c^*(A)$ derived in [18] and stated in Theorem 4.3 above. An analogous characterization for spreading speeds was given by Gärtner and Freidlin [32] based on probabilistic arguments.

Henceforth, we denote:

$$m = f'(0) > 0, \quad \Delta = \Delta_y$$

To analyze the previous eigenvalue problem (4.7), we define, for all $t \geq 0$, $\mu^c(t)$ as the principal (first) eigenvalue of the problem.

$$\begin{cases} -\Delta\varphi + t(c + A\alpha(y))\varphi - m\varphi = \mu^c(t)\varphi & \text{in } \omega \\ \varphi > 0 & \text{in } \omega, \quad \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\omega \end{cases} \quad (5.2)$$

Recall that the sign condition $\varphi > 0$ of the eigenfunction characterizes $\mu(t)$ as the first eigenvalue. With these notations, clearly problem (4.7) is equivalent to solving the equation:

$$\mu^c(t) = t^2, \quad t > 0 \quad (5.3)$$

To prove (ii) and (iii) in Theorem, we define:

$$s = At \quad \text{and} \quad \gamma = \frac{c}{A}$$

Then define $\nu^\gamma(s)$ as the principal eigenvalue of the problem:

$$\begin{cases} -\Delta\varphi + (\gamma + \alpha(y))s\varphi - m\varphi = \nu^\gamma(s)\varphi & \text{in } \omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\omega. \end{cases} \quad (5.4)$$

With this change of variables, it is clear that problem (5.3) is equivalent to the equation:

$$\nu^\gamma(s) = \frac{s^2}{A^2} \quad (5.5)$$

The properties of $\nu^\gamma(s)$ are the same as those of $\mu^c(t)$ (see [18]) and are listed in the following proposition:

Proposition 5.1 .

- (i) $\nu^\gamma(s)$ is a continuous and concave function of s .
- (ii) For $\gamma = 0$, $\nu^0(s) \leq -m$, $\forall s \geq 0$.
- (iii) $\nu^\gamma(s) - \nu^{\gamma'}(s) = s(\gamma - \gamma')$, for all s, γ, γ' .

Property (iii) is obvious. To prove (ii), consider the eigenfunction φ corresponding to $\nu = 0$:

$$\begin{cases} -\Delta\varphi + \alpha(y)s\varphi = (\nu^0(s) + m)\varphi & \text{in } \omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\omega; \end{cases} \quad (5.6)$$

Since $\varphi > 0$ in $\bar{\omega}$ (by the strong maximum principle) we may divide this equation by φ to get:

$$\nu^0(s) + m = s \int_{\omega} \alpha(y) dy - \int_{\omega} \frac{|\nabla \varphi|^2}{\varphi^2} = - \int_{\omega} \frac{|\nabla \varphi|^2}{\varphi^2} \leq 0$$

Hence, since $\alpha(y) \not\equiv 0$, the inequality is strict and we get $\nu^0(s) < -m$.

The other properties follow from the variational characterization of $\nu^\gamma(s)$:

$$\nu^\gamma(s) = \min_{\varphi \in H^1(\omega)} \left\{ \frac{\int_{\omega} |\nabla \varphi|^2 + (\gamma + \alpha(y)) s \varphi^2 - m \varphi^2}{\int_{\omega} \varphi^2} \right\}$$

As a function of s , the argument of this minimum is affine, and therefore $\nu^\gamma(s)$ is concave (hence continuous).

From the properties in the above proposition, it follows that for all $A > 0$, there exists a value $\gamma^*(A)$ such that:

$$\nu^\gamma(s) < \frac{s^2}{A^2}, \quad \text{for all } s > 0 \quad \text{if } \gamma < \gamma^*(A), \quad (5.7)$$

$$\text{there exists } s > 0 \quad \text{with} \quad \nu^\gamma(s) = \frac{s^2}{A^2} \quad \text{for all } \gamma \geq \gamma^*(A) \quad (5.8)$$

More precisely, $\nu^\gamma(s) = \frac{s^2}{A^2}$ has a unique solution $s > 0$ if $\gamma = \gamma^*(A)$ and it has exactly two solutions if $\gamma > \gamma^*(A)$.

Clearly, one has:

$$\gamma^*(A) = \frac{c^*(A)}{A} \quad (5.9)$$

Since $\frac{s^2}{A^2} < \frac{s^2}{A'^2}$ if $A' < A$, it is now obvious that $\gamma^*(A') > \gamma^*(A)$. Therefore, $c^*(A)/A$ is decreasing as $A > 0$ increases which proves (ii) in the Theorem.

Furthermore, since $\nu^0(1) < -m$, by continuity, we see that there exists $\delta > 0$ such that for all $0 \leq \gamma \leq \delta$,

$$\nu^\gamma(1) \leq -m \quad (5.10)$$

and

$$\nu^\gamma(s) \leq -\frac{m}{2} \quad \text{for all } s \in [0, 1] \quad (5.11)$$

Hence, for all $0 \leq \gamma \leq \delta$, since $\nu^\gamma(0) = -m$, from (5.10) and by the concavity of $\nu^\gamma(s)$, we see that $\nu^\gamma(s) \leq -m, \forall s \geq 1$. Together with (5.11) this shows that:

$$\nu^\gamma(s) \leq -\frac{m}{2} < 0, \quad \forall s \geq 0, \quad \forall \gamma \in [0, \delta] \quad (5.12)$$

Consequently, we have:

$$\gamma^*(A) \geq \delta, \quad \forall A \geq 0 \quad (5.13)$$

Therefore,

$$\lim_{A \nearrow +\infty} \searrow \frac{c^*(A)}{A} = \rho > 0 \quad (5.14)$$

which proves part (iii) in Theorem 5.1.

It now remains to show property (i) in the theorem, namely that $c^*(A)$ is increasing with $A > 0$. To this end, we use the original formulation with variables c and t . We denote by $\mu^A(t)$ and by φ_A the principal eigenvalue and associated eigenfunction of the problem:

$$\begin{cases} -\Delta \varphi_A + t(c + A \alpha(y))\varphi_A - m\varphi_A = \mu^A(t)\varphi_A & \text{in } \omega, \\ \frac{\partial \varphi_A}{\partial \nu} = 0 & \text{on } \partial\omega \end{cases} \quad (5.15)$$

The eigenfunction φ_A is uniquely determined by the normalization:

$$\int_{\omega} \varphi_A^2(y) dy = 1 \quad (5.16)$$

We recall that the property $\varphi_A > 0$ characterizes the principal eigenvalue.

From Krein-Rutman theory, we know that $\mu^A(t)$ is simple. It is then well known that φ_A and $\mu^A(t)$ are differentiable functions of the parameter A .

Let us denote $\psi = \frac{\partial \varphi}{\partial A}$ (to shorten notations we write $\psi = \psi_A(y)$). Differentiating equation (5.15) with respect to A , we obtain:

$$\begin{cases} -\Delta \psi + (c + A \alpha(y)) t \psi & -m\psi + \alpha(y) t \varphi_A \\ & = \mu^A(t) \psi + \frac{\partial \mu^A(t)}{\partial A} \varphi_A \quad \text{in } \omega, \\ \frac{\partial \psi_A}{\partial \nu} & = 0 \quad \text{on } \partial \omega. \end{cases} \quad (5.17)$$

By (5.16), we also have:

$$\int_{\omega} \psi \varphi_A = 0 \quad (5.18)$$

Multiplying (5.15) by ψ and (5.17) by φ_A , integrating and using Green's formula, we derive:

$$\int_{\omega} \alpha(y) t \varphi_A^2 = \left(\frac{\partial \mu^A(t)}{\partial A} \right) \int_{\omega} \varphi_A^2 \quad (5.19)$$

Let us show that the left hand side of (5.19) is negative.

By the variational characterization of $\mu^A(t)$ in (5.15), letting:

$$R(\varphi) = \frac{\int_{\omega} |\nabla \varphi|^2 + t \int_{\omega} (c + A \alpha(y)) \varphi^2}{\int_{\omega} \varphi^2}$$

we know that:

$$\mu^A(t) + m = \min_{\varphi \in H^1(\omega)} R(\varphi) \quad (5.20)$$

Hence, $\mu^A(t) + m < R(1)$, since 1 is not an eigenfunction, as $\alpha \not\equiv 0$. Now,

$$R(1) = c t + A t \int_{\omega} \alpha(y) = c t.$$

Thus, we derive:

$$\mu^A(t) + m = \int_{\omega} |\nabla \varphi|^2 + c t + A t \int_{\omega} \alpha(y) \varphi^2 < c t,$$

and this proves that for $t > 0$,

$$\int_{\omega} \alpha(y) \varphi^2 < 0. \quad (5.21)$$

Therefore (5.19) shows that:

$$\frac{\partial \mu^A(t)}{\partial A} < 0, \quad \text{for } t > 0. \quad (5.22)$$

Now, let us take two values A, A' with:

$$0 < A < A'.$$

By (5.22) we see that:

$$\mu^A(t) > \mu^{A'}(t), \quad \forall t > 0$$

Recall that $\mu^A(t)$ also depends on c and write $\mu^A(t) = \mu_c^A(t)$.

Since, for $c = c^*(A)$, the function $\mu_{c^*(A)}^A(t)$ is tangent to the curve t^2 at exactly one point, the previous inequality implies that:

$$\mu_{c^*(A)}^{A'}(t) < t^2, \quad \forall t \geq 0 \quad (5.23)$$

Then, by the definition of $c^*(A')$ (see Theorem 4.3), we finally establish that:

$$c^*(A') > c^*(A), \quad (5.24)$$

which prove, that $A \rightarrow c^*(A)$ is an increasing function.

This completes the proof of Theorem 1.

Remark. If, instead of a KPP type nonlinearity, we consider a combustion type nonlinearity f , then, there exists a unique value $c(A)$ for all A for which there is a traveling front solution of (1). In this case, by comparison, it is easy to get an upper bound $c(A) \leq kA$ for some constant k . Recently, lower bounds of the type $c(A) \geq k_0 A$ for some constant $k_0 > 0$ have been established by Constantin, Kiselev and Ryzhik [23]. This lower bound proves that the enhancement by advection is at least linear. The proof of [23] uses of very delicate estimates on the parabolic equation. Another interesting proof was also given by S. Heinze [39].

It is an open problem in this case, to know whether $c(A)$ has the same properties as those of $c^*(A)$ in Theorem 1.

6 Asymptotic speed of spreading with shear flows in cylindrical domains

In the same geometry as above, with the same shear flow parallel to the walls of the cylinders, let us now consider the initial value problem with Neuman conditions

$$\begin{aligned} u_t - \Delta u + \alpha(y)u_x &= f(u), & x \in \Sigma \\ u_\nu &= 0, & (x, y) \in \mathbb{R} \times \partial\omega \\ u(0, x) &= u_0(x) \end{aligned} \quad (6.1)$$

The function $\alpha(y)$ is taken as given. Again the discussion is done according to the type of nonlinearity.

6.1 KPP case

Assume first that f is of KPP type, that is satisfies condition (4.6). The traveling waves have this time the form: $\phi(x_1 + ct, y)$. Let c_* : minimal speed of the waves propagating from right to left as above. Similarly, there is a minimal speed c_{**} for the the waves propagating from left to right. Denote by ϕ_{c_*} the front solution of (4.3) with minimal speed such that $\phi_{c_*}(0, y_0) = 1/2$ for some fixed point $y_0 \in \omega$.

Theorem 6.1 (Freidlin [31], Mallordy-Roquejoffre [51], 1995)

1 (Convergence to TF). *Assume that the initial data $u_0(x_1, y)$ is the Heaviside function $H(x_1)$. Then*

$$\lim_{t \rightarrow +\infty} u(t, x_1 + ct, y) = \begin{cases} 1 & \text{if } c < c_* \\ 0 & \text{if } c > c_* \end{cases}$$

2 (Spreading). *Assume $u_0(x_1, y)$ to be compactly supported. Then*

$$\lim_{t \rightarrow +\infty} u(t, x_1 + ct, y) = \begin{cases} 1 & \text{if } c_{**} < c < c_* \\ 0 & \text{if } c \notin [c_{**}, c_*] \end{cases}$$

As in the one dimensional setting, the speed c_* is the speed of TF which is selected for rapidly decreasing initial data. The proof in [51] is with PDE arguments and relies in particular on the Maximum principle while the approach of Freidlin [31] is probabilistic.

One may always define a function $\xi(t, y)$ such that $u(t, \xi(t, y), y) = 1/2$. A sufficient condition for this is that u_0 should be nondecreasing in x_1 . For all initial data giving rise to the behavior of case 1 above, we have $\xi(t, y) =$

$c_*t + o(t)$ as $t \rightarrow \infty$. It is an open problem to find the next term in the expansion with a rigorous proof.

A stability result for traveling fronts with speeds higher than critical is also available here.

Theorem 6.2 (Mallordy-Roquejoffre [51], 1995)

Choose $c > c_*$ and suppose $\frac{u_0}{\partial_{x_1}\phi_c}$ decays exponentially as $|x_1| \rightarrow +\infty$. Then

$$u(t, x_1, y) - \phi_c(x_1 + ct) = O(e^{-\delta t}) \quad \text{for some } \delta > 0.$$

Note that here too, there is no spatial shift. The assumption at $x_1 = +\infty$ can be considerably relaxed (see [51]).

6.2 Combustion case

Assume here that f satisfies condition(4.4). Denote by (c, ϕ) the unique traveling front solution of (4.3) such that $\phi(0, y_0) = \theta$ for some $y_0 \in \omega$.

Theorem 6.3 (Roquejoffre [58], 1997). Assume $u_0(x_1, y) - H(x_1)$ to have compact support. There exists $a \in \mathbb{R}$ such that

$$|u(t, x_1, y) - \phi(x_1 + ct + a, y)| = O(e^{-\delta t})$$

for some $\delta > 0$, uniformly in (x, y) .

Here again, the conditions at $+\infty$ can be relaxed. The proof relies on two basic properties.

(i) Exponential stability property :

Theorem 6.4 (Berestycki-Larrouturou-Roquejoffre [14], 1991).

Assume $u_0(x_1, y) - H(x_1)$ to have compact support and to be sufficiently small. Then there exists $a \in \mathbb{R}$ such that

$$|u(t, x_1, y) - \phi(x_1 + ct + a, y)| = O(e^{-\delta t})$$

for some $\delta > 0$, uniformly in (x_1, y) .

(ii) An asymptotic monotonicity property:

Theorem 6.5 Given a compact subset of $\bar{\Sigma}$, the function $t \mapsto \partial_{x_1} u(t, x_1, y)$ becomes positive in finite time on K .

In a sense, this can be viewed as the parabolic analogue of some monotonicity results in [17] for elliptic equations in cylinders.

6.3 Bistable case

Consider now an f of the bistable type, that is satisfying condition (4.9). Assume the existence of a traveling front solution of (4.3). As we have seen, a sufficient condition for this is that ω is convex. Uniqueness of the fronts (c, ϕ) however always hold – without assumption on ω . Let ϕ be the unique front such that $\phi(0, y_0) = \theta$.

Theorem 6.6 (Roquejoffre [58], 1997).

Assume:

$$\limsup_{x_1 \rightarrow -\infty} u_0(x_1, y) < \theta, \quad \liminf_{x_1 \rightarrow +\infty} u_0(x_1, y) > \theta$$

uniformly in y . Then there exists $a \in \mathbb{R}$ such that

$$|u(t, x_1, y) - \phi(x_1 + ct + a, y)| = O(e^{-\delta t})$$

for some $\delta > 0$, uniformly in (x_1, y) .

The tools are the same as above.

One may also prove spreading results in the cases of bistable or combustion nonlinearities, extending the one dimensional results. These are of the following type: Assume that u_0 is compactly supported, and that the set $\{u_0 \geq \theta\}$ is large enough. Then the solution $u(t, \cdot)$ converges exponentially to a pair of fronts traveling in opposite directions.

7 Pulsating traveling fronts

With the important exception of equations involving transport by a parallel flow of sections 4 and 5, the notion of traveling front is essentially restricted to the homogeneous equation (1.1) set in all of space. I shall now describe a generalization which is relevant for general periodic settings. Indeed, in order to understand propagation in heterogeneous media, one usually considers either periodic media or random media. I shall not discuss here the latter. As was already pointed out, many works concerning random media are reviewed in the paper by J. Xin [74] where one will find references to the literature on the subject.

The spatially heterogeneous character of the problem may be related either to the geometry or to the equation or to both.

To start with the geometry, consider again the homogeneous equation but now set in a domain Ω :

$$u_t - \Delta u = f(u) \quad \text{in } \Omega \quad (7.1)$$

with Neumann boundary conditions:

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \quad (7.2)$$

Such a problem arises for instances when Ω is the whole space with a periodic array of holes. More precisely, let G be a compact set of $\mathbb{R}^n$ with smooth boundary. Assume that for some $L_1, \dots, L_n > 0$, the family of translates of G by vectors $(k_1 L_1, \dots, k_n L_n)$ with integers $k_1, \dots, k_n$ are pairwise disjoint.

Let:

$$H = \cup_{k_1, \dots, k_n \in \mathbf{Z}} \{G + k_1 L_1 \vec{e}_1 + \dots + k_n L_n \vec{e}_n\} \quad (7.3)$$

$(\vec{e}_1, \dots, \vec{e}_n)$ is the canonical basis of $\mathbb{R}^n$) and define:

$$\Omega = \mathbb{R}^n \setminus H$$

Then, it is easily seen that there are no traveling front solutions (in the sense of section 1) to equations (7.1)-(7.2). One is thus led to extend this notions. In this framework, one defines the notion of *pulsating traveling front* solution – which I write as PTF solution, for short, in the following.

I first give the definition for a front propagating in the $-\vec{e}_1$ direction.

Definition 7.1 . *A PTF solution propagating in the direction $-\vec{e}_1$ is a solution $u(t, x)$ of (7.1) - (7.2) defined for all time $t \in \mathbb{R}$ with the following properties.*

1. $u(t, x_1, y) \rightarrow 0$ as $x_1 \rightarrow -\infty$
 $u(t, x_1, y) \rightarrow 1$ as $x_1 \rightarrow +\infty$
with the limits being uniform with respect to $y = (x_2, \dots, x_n)$.
2. $u(t, x_1, x_2, \dots, x_n)$ is L_i -periodic in the variables x_i for $i = 2, \dots, n$.
3. There exists $T > 0$ such that $u(t + T, x_1, y) = u(t, x_1 + L_1, y)$ for all $t \in \mathbb{R}, (x_1, y) \in \Omega$.

The ratio $c = \frac{L_1}{T}$ is then called the average speed of the front.

Travelling fronts are solutions with a profile which is constant and moves at a given constant velocity. Here, in this more general setting, the profile of the solution is periodic in time (with period T) and moves with an average speed which is the same over each period.

The same definition applies to different periodic geometries. Consider for instance the case when Ω is a cylinder with oscillating boundary. That is, suppose that Ω is a smooth domain, bounded in directions $x_2, \dots, x_n$, and:

$$\Omega = \{x = (x_1, y) \in \mathbb{R}^n, \quad y \in \omega(x_1)\}$$

where $\omega(x_1)$ is a bounded open set in $\mathbb{R}^{n-1}$ which is L_1 -periodic with respect to the variable x_1 . In this framework, a PTF solution is a globally defined solution which satisfies properties 1 and 3 in the above definition.

Going back to the setting of a periodic array of holes (7.3) above, one can consider more generally a PTF propagating in the direction $-\vec{e}$ where $\vec{e}$ is a unit vector $|\vec{e}| = 1$. Here is the definition:

Definition 7.2 . *A PTF solution propagating in the direction $-\vec{e}$ in a globally (in time) defined solution $u(t, x)$ with the following properties:*

1. $u(t, x) \rightarrow 0$ as $x \cdot \vec{e} \rightarrow -\infty$
 $u(t, x) \rightarrow 1$ as $x \cdot \vec{e} \rightarrow +\infty$
with limits uniform with respect to $y = x - (x \cdot \vec{e})\vec{e}$.
2. *There exists $c > 0$, called the average speed of the front such that*
 $u(t + L_i \frac{\vec{e} \cdot \vec{e}_i}{c}, x) = u(t, x + L_i \vec{e}_i), \quad \text{for all } i = 1, \dots, n.$

Note that in the case $\vec{e} = \vec{e}_1$, property 2 in this definition covers both properties 2 and 3 in the first definition 7.1.

This notion is an extension of that of traveling fronts for in the case of the homogeneous space $\Omega = \mathbb{R}^N$, PTF are traveling fronts. Indeed, consider a solution u of (7.1), which is, say a PTF in direction $-\vec{e}_1$. Since in this case $L_2, \dots, L_n$ are arbitrary numbers, u is actually independant of $x_2, \dots, x_n$. Furthermore, applying property 3 for all $L_1 > 0$ one sees that for all t there exists $\gamma(t)$ such that:

$$u(t, x) = u(0, x + \gamma(t) \vec{e}_1)$$

It is easily seen then that $\dot{\gamma}(t)$ is constant. That is, after a shift, $u(t, x) = \phi(x_1 + ct, x_2, \dots, x_n)$ for some function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$. Hence u is a classical traveling front.

To give a definition in a very general setting, one considers a domain Ω which is periodic with respect to d variables $x_1, \dots, x_d$ and bounded in the remaining variables $x_{d+1}, \dots, x_n$ ($1 \leq d \leq n$). The case of holes corresponds to $d = n$ and that of an undulating cylinder to $d = 1$. The definition 7.2 needs only to be modified by requiring that property 2 holds for all $i = 1, \dots, d$ and otherwise applies as such for PTF in such a domain. In the following, a periodic domain Ω will always refer to this situation.

As said before, spatial heterogeneity also arises from equations which are not homogeneous. A general class of equations is that of the form:

$$u_t - \nabla \cdot (A(x) \nabla u) + \vec{q}(x) \cdot \vec{\nabla} u = f(x, u) \quad \text{in } \mathbb{R}^n. \quad (7.4)$$

Assume that the matrix $A(x)$ is self adjoint, uniformly elliptic, and that $A(x)$, $\vec{q}(x)$ and $f(x, u)$ are functions which are periodic in the variables $x_1, \dots, x_n$ with respective periods $L_1, \dots, L_n$.

Here again, the notion of traveling front does not apply and is generalized to that of PTF. Definition 2 applies *verbatim* to characterize such solutions of (7.4) propagating in a direction $\vec{e}$.

The notion of PTF was actually introduced in this setting by Shigesada and Kawasaki [62]. With the motivation to understand the effects of spatial heterogeneity on biological invasions, they considered an equation of the type:

$$u_t - (d(x) u_x)_x = f(x, u), \quad x \in \mathbb{R}$$

in one space dimension. This model, which is also pertinent to discuss conservation of species (see [62]), is the so-called patch model. It describes the effects of an environment made of alternating favorable and less favorable patches. These authors introduced the notion of PTF to identify biological invasions in a periodic setting. We refer the reader to [62] and [41] for a study of this model and to [12] for very recent progress on this type of models.

More generally, one can consider the equation set in a domain Ω :

$$u_t - \nabla \cdot (A(x) \nabla u) + \vec{q}(x) \cdot \vec{\nabla} u = f(x, u) \quad \text{in } \Omega \quad (7.5)$$

with boundary conditions (7.2). For a general domain Ω as above, periodicity of $A(x), \vec{q}(x), f(x, u)$ refers to the variables $x_1, \dots, x_d$. Again definition 2 applies as it stands to describe PTF solutions of (7.5) with boundary condition (7.2) in this general setting.

In the context of PTF solutions, there is a change of variables analogous to that of traveling fronts described in sections 4 and 5 which has been introduced by J. Xin [70], see also the papers [69, 71, 72, 73] by J. Xin. Indeed, define:

$$u(t, x) = \phi(x \cdot \vec{e} + ct, x) \quad (7.6)$$

and set $s = x \cdot \vec{e} + ct$; $\phi = \phi(s, x)$ is defined for $s \in \mathbb{R}$ and $x \in \Omega$. Equation (7.1) is transformed into:

$$c \phi_s - \phi_{ss} - \Delta_x \phi - 2 \sum_{i=1}^n \phi_{s, x_i} e_i = f(x, \phi) \quad \text{in } \Omega \quad (7.7)$$

and (7.2) yields

$$\nabla_x \phi \cdot \vec{\nu} + (\vec{\nu} \cdot \vec{e}) \phi_s = 0 \quad \text{on } \partial\Omega \quad (7.8)$$

In contradistinction with traveling fronts where the changes of variables reduce the number of variables, here ϕ is again a function of $n + 1$ variables. Hence, the problem (7.7) that one obtains is *not* elliptic but rather degenerate elliptic. In other terms, the parabolic character is preserved under this change of variables. This degenerate character is the source of new technical difficulties which are not encountered when dealing with classical traveling fronts.

For the general equation (7.5) - (7.2), one also easily derives the equation satisfied by ϕ (see [9]). Since the primary interest in this paper is the problem of combustion with transport, I will restrict myself to write the equation in this setting. Starting from the equation:

$$u_t - \Delta u + \vec{q}(x) \cdot \vec{\nabla} u = f(u) \quad \text{in } \Omega \quad (7.9)$$

one obtains:

$$\begin{cases} -\Delta_{s,x} \phi - 2 \sum_{i=1}^d \phi_{s, x_i} e_i + \vec{q}(x) \cdot \vec{\nabla}_x \phi + (c + \vec{q}(x) \cdot \vec{e}) \phi_s = f(\phi), \\ \text{for } s \in \mathbb{R}, x \in \Omega \end{cases} \quad (7.10)$$

Lastly, PTF solutions propagating in a tube have been observed in some beautiful experiments by P. Ronney [56].

Even for shear flows described in Section 4 above, the notion of PTF is relevant in most cases. Indeed, consider again the equation:

$$u_t - \Delta u + \alpha(y) \frac{\partial u}{\partial x_1} = f(u) \quad \text{in } \mathbb{R}^n \quad (7.11)$$

with $y = (x_2, \dots, x_n)$ and α a continuous and periodic function in all variables on $\mathbb{R}^{n-1}$. This equation (7.11) admits TF solutions in direction $\vec{e}_1$, namely the ones described in Section 4. It also has TF solutions in any direction $\vec{e}$ orthogonal to $\vec{e}_1$. Indeed, the advection term then drops out and it is sufficient to look for planar fronts (see Sections 2, equation 2.3) of the form $u(t, x) = \phi(x \cdot \vec{e} + ct)$ for such a direction. It is simple to see that these two kinds are the only traveling fronts in this case. Indeed, in [9], we prove that for any direction $\vec{e}$ not orthogonal to $\vec{e}_1$, and such that $\vec{e} \neq \pm \vec{e}_1$, there are no TF solutions if α is not constant. In such directions however, PTF are relevant and the existence results of the next section also apply here.

8 Existence of pulsating traveling fronts

The theory of existence of traveling fronts has recently been extended to PTF solutions in periodic settings in almost full generality for the cases of combustion or positive nonlinear-reaction terms f . For the sake of simplicity, I shall assume here that neither f nor the diffusion depend on the variable x . Since the interest here rests primarily on transport and on geometry, the equation involves an advection term and is set in a general domain Ω .

That is, I consider the equation (7.9) above. The results are similar for more general equations involving nonhomogeneous diffusion and reaction (*i.e.* $f = f(x, u)$). For precise statements, the reader is referred to [9].

Only but few perturbation results are available in the bistable case. It is known that there is not always existence (fronts may be “blocked” in this case). I refer the reader to the works [39, 52, 54, 55, 72] for this case.

8.1 PTF for combustion and KPP nonlinearities

For the remaining cases, as in the homogeneous case, one has to distinguish between combustion nonlinearities and positive nonlinearities.

I now state the results for both cases. Throughout this section, it is assumed that $\Omega \subset \mathbb{R}^n$ is a smooth domain which is periodic in the variables $(x_1, \dots, x_d)$ and bounded in the variables $y = (x_{d+1}, \dots, x_n)$. The advection term $\vec{q}(x)$ is assumed to be periodic with respect to $(x_1, \dots, x_d)$.

The advection vector field $\vec{q}(x)$ is assumed to be C^1 and satisfies:

$$\vec{\nabla} \cdot \vec{q} = 0 \quad \text{in } \Omega \quad (8.1)$$

$$\vec{q} \cdot \vec{\nu} = 0 \quad \text{on } \partial\Omega \quad (8.2)$$

$$\int_{\mathcal{C}} q_i(x) dx = 0, \quad \text{for } i = 1, \dots, d, \quad (8.3)$$

where $\vec{\nu}$ stands for the unit outward normal on $\partial\Omega$ and $\mathcal{C}$ is a periodicity cell of Ω .

Theorem 8.1 *In the above setting, consider a nonlinear term $f(u)$ of combustion type, i.e. satisfying assumptions (4.4). Then, there exists a unique solution (c, ϕ) of (7.9 and 7.2), that is, u is unique up to shifts of time. Moreover, $c > 0$ and $\frac{\partial \phi}{\partial t} > 0$.*

This result was first proved by J. Xin [69], [71] in the case $\Omega = \mathbb{R}^n$ by using an elaborate continuation method. In a general geometry, this is proved in [9]. There, we also prove the same result for a general equation 7.5 assuming $A(x)$ and $f(x, u)$ to be periodic in $(x_1, \dots, x_d)$ under some conditions on $f(x, u)$.

Let me turn next to the positive nonlinearity case.

Theorem 8.2 *Consider the same geometric setting as above, and assume that f is positive in $(0, 1)$. Then, there exists a critical speed c^* such that:*

- (i) *For all $c \geq c^*$, there exists a PTF solution of (7.9) and (7.2) with speed c and such that $\frac{\partial \phi}{\partial t} > 0$.*
- (ii) *No PTF solution of (7.9) and (7.2) exists with speed $c < c^*$.*

Moreover, if one assumes that $f'(0) > 0$, then any solution (c, ϕ) of (7.9) and (7.2) satisfies $\frac{\partial \phi}{\partial t} > 0$ for all $t \in \mathbb{R}, x \in \Omega$.

This Theorem is established with François Hamel in [9]. The only known earlier result for this case is due to Hudson and Zinner [41] for a one dimensional problem, with no advection ($\vec{q} = 0$) and a particular class of functions $f(x, u)$ periodic in x . In [9], we also treat the general equation case (7.5) with suitable assumptions on $f(x, u)$.

The monotonicity property in time is proved in [9] using a variant of the sliding method of [17] that is adapted for parabolic equations. The novelty is that we use translations in time, rather than translations in space. In the case $f'(0) = 0$, it is an open problem to know whether any solution of (7.9) and (7.2) satisfies this property. It should also be emphasized that the uniqueness of solutions (c, ϕ) of (7.9) and (7.2) with $c \geq c^*$ is an open question.

In the case of KPP nonlinearities, a formula can further be obtained for the critical speed c^* . This is stated in the next result which is joint work with F. Hamel and N. Nadirashvili [10].

It relies one again on the linearized equation about $u = 0$:

$$u_t - \Delta u + \vec{q}(x) \cdot \vec{\nabla} u - f'(0)u = 0 \quad (8.4)$$

Looking for *exponential solutions* in the direction $\vec{e}$ now involves solution of (8.4) of the type:

$$u(t, x) = e^{\lambda(x \cdot \vec{e} + ct)} \varphi(x)$$

With respect to the previous cases (traveling fronts) φ now depends on all variables. The requirement is that φ be periodic with respect to the variables $x_1, \dots, x_d$.

Thus, one is led to the following “eigenvalue” problem in λ and φ :

$$\left\{ \begin{array}{ll} -\Delta \varphi - 2\lambda \vec{\nabla} \varphi \cdot \vec{e} + \vec{q}(x) \cdot \vec{\nabla} \varphi \\ \quad + [\lambda c + \lambda \vec{q}(x) \cdot \vec{e} - f'(0)] \varphi = \lambda^2 \varphi & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial \nu} + \lambda(\vec{e} \cdot \vec{\nu}) \varphi = 0 & \text{on } \partial \Omega, \\ \varphi(x) \text{ is periodic in } x_1, \dots, x_d & . \end{array} \right. \quad (8.5)$$

Theorem 8.3 [10]. *In the above geometrical setting, under assumptions (8.1) - (8.3) on $\vec{q}(x)$, let f satisfy condition (4.6). Then, the critical speed c^* is characterized in terms of the spectral problem (8.5) by the following property:*

- (i) *For $c < c^*$, there is no solution of (8.5) with $\lambda > 0$ and $\varphi > 0$,*

- (ii) For $c = c^*$, there is one and only one “eigenvalue” $\lambda > 0$ of (8.5) corresponding to an eigenfunction $\varphi > 0$,
- (iii) For all $c > c^*$, there are exactly two values of $\lambda > 0$ for which there is an eigenfunction $\varphi > 0$ of (8.5).

Even though problem (8.5) appears to be somewhat intricate, much information on the critical speed c^* can be gathered from this theorem as will be seen in section 9 below.

Some remarks about (8.5) and this characterization of c^* are in order. Firstly, because φ is sought as a periodic function, problem (8.5) is a compact problem. It suffices to consider it on a periodicity cell $\mathcal{C}$ of Ω .

Next, when there is no transport terms *i.e.* $\vec{q}(x) = 0$, and $\Omega = \mathbb{R}^n$ then the principal eigenfunction φ is clearly a constant and 8.5 reduces to equation (4.8) above. This further holds true when $\vec{q}(x) = 0$, Ω is a cylinder $\Omega = \mathbb{R} \times \omega$ and $\vec{e} = \vec{e}_1$. Note, however, that even when $\vec{q}(x) = 0$, that is, for the homogeneous reaction-diffusion equation, this is not the case any more when Ω is not a cylinder with axis in the $\vec{e}$ direction.

In the particular case of a shear flow $\vec{q}(x) = \alpha(y) \vec{e}_1$ with $y = (x_2, \dots, x_n)$, then the exponential solutions take the particular form with separation of variables:

$$u = e^{\lambda(x_1 + ct)} \varphi(y)$$

Clearly, in this case, problem 8.5 reduces to 4.7 above.

But in general, one has to analyse the full spectral problem 8.5.

8.2 Asymptotic speed of spreading in periodic media

Gärtner and Freidlin [32] have established the asymptotic speed of spreading for equations with periodic diffusion coefficients in the KPP case. These have been further extended by Freidlin [29] to equations involving a periodic drift term (or advection) $\vec{q}(x)$ as above. The results are stated in $\mathbb{R}^N$.

Theorem 8.4 [32], [29]. *Assume that f satisfies condition (4.6) and that the matrix $A(x)$ and the field $\vec{q}(x)$ are L_i -periodic in the variable x_i for all $i = 1, 2, \dots, N$. Assume further that $A(x)$ is bounded from below by $c_0 I$ with $c_0 > 0$ and that $\vec{q}(x)$ satisfies the assumptions (8.1)–(8.3) above.*

For each unit vector $\vec{e}$ in $\mathbf{R}^N$, let $c^*(\vec{e})$ denote the positive number characterized by the eigenvalue problem (8.5) (see Theorem 8.3).

Let u_0 be a continuous function, $0 \leq u_0 \leq 1$, $u_0 \not\equiv 1$ with compact support. Let $u(t, x)$ denote the solution of (7.4) with initial datum $u(0, x) = u_0(x)$. Then, for all $\vec{e}$, $|\vec{e}| = 1$, the following limits hold:

(i) for all $v < c^*(\vec{e})$:

$$\lim_{t \nearrow +\infty} \inf_{|x| \leq vt} u(t, x) = 1$$

(ii) for all $v > c^*(\vec{e})$:

$$\lim_{t \nearrow +\infty} \sup_{|x| \geq vt} u(t, x) = 0$$

The results in [29], [32] involve a somewhat different formulation than (8.5). But, arguing as in J. Xin [74], the two formulations can be shown to be equivalent.

Thus, in a periodic heterogeneous environment, the spreading speed coincides with the critical speed of traveling fronts. In contradistinction with the homogeneous case, it may vary with the direction $\vec{e}$.

The proof of Gärtner and Freidlin [32] and Freidlin [29] are probabilistic. Recently, a new analytical approach to these questions has been proposed by H. Weinberger [66, 67]. A complete PDE proof of this Theorem is still open. Note that the articles [29], [32] do not deal with the existence of propagating fronts but with the spreading property.

Thus, the theory of KPP in the homogeneous case, is fully generalized to propagation in periodic media.

Some further estimates on the speed of propagation in a more general setting are given in a forthcoming joint work with F. Hamel and N. Nadirashvili [10].

9 Asymptotics for large advection in periodic vortical flows

From the point of view of turbulent combustion, it is important to understand the effect of vortices in the flow on the propagation of fronts. (See

e.g. [21, 75]). With this aim, I now consider a flow with a periodic array of vortical cells and examine the influence of large amplitude on the speed of propagation of PTF.

The discussion here is restricted to dimension $N = 2$. Since the flow is divergence free, we may assume that there is a function $\psi = \psi(x_1, x_2)$ such that:

$$q_1 = -\frac{\partial\psi}{\partial x_2}, \quad q_2 = \frac{\partial\psi}{\partial x_1}$$

For simplicity, let us consider a strip:

$$\Sigma = \mathbb{R} \times (-h, h)$$

It is assumed that ψ is periodic in x_1 . We shall be looking at PTF propagating in the direction x_1 .

As we have seen in the previous section, if f is of combustion type, for all $A \geq 0$, there exists a unique speed $c(A)$ and a unique PTF solution of the equation:

$$\begin{cases} u_t - \Delta u + A \vec{q}(x) \cdot \vec{\nabla} u = f(u) & \text{in } \Sigma, \\ \frac{\partial u}{\partial \nu}(t, x) = 0 & \text{on } \partial\Sigma. \end{cases} \quad (9.1)$$

If f is of the KPP type, then there is a minimal speed $c^*(A)$ for the PTF solutions of (9.1). The question here is to determine the behavior of $c(A)$ or $c^*(A)$ for large A .

As a model for flame propagation, it is customary to take $f(u)$ of the form:

$$f(u) = \frac{1}{\epsilon} \phi\left(\frac{u-1}{\epsilon}\right)$$

with $\int_{-\infty}^0 \phi(s) ds < \infty$, $\phi > 0$. Then, taking the limit as $\epsilon \rightarrow 0$ (“high activation energy asymptotics”), one derives a free boundary problem for u . Such a limiting procedure is carried out for the shear flow case in [7] but is still only formal in the case of more general flows. At any rate, for small $\epsilon > 0$, at least formally, the “flame thickness” becomes small.

In joint work with B. Audoly and Y. Pomeau [3], we have considered this case, assuming the flame thickness to be much smaller than the typical length scales of the flow. Using formal arguments, relying in particular on a fine

analysis of the scales of the problem, we derive the following.

Proposition 9.1 (B. Audoly, H. Berestycki, Y. Pomeau [3], 2000) *In the above setting, when $A \rightarrow \infty$, the unique speed $c(A)$ of PTF has the behavior*

$$c(A) \sim c_0 A^{1/4} \quad (9.2)$$

for some constant $c_0 > 0$.

The analysis in [3] relies on examining separately the issues of propagation in each vorticity cell – where u is approximately constant on level sets of ψ – and the transition from one cell to the next one.

It is still an open problem to establish formula 9.2 in a rigorous way. Very recently, A. Kiselev and L. Ryzhik have established a lower bound.

Theorem 9.1 (A. Kiselev, L. Ryzhik [45], 2002). *The speed $c(A)$ in the combustion case, or the critical speed $c^*(A)$ in the KPP case satisfy the lower bound:*

$$c(A) \geq C_1 A^{1/5} \quad \text{as } A \rightarrow \infty \quad (9.3)$$

for some constant $C_1 > 0$

The proof of (9.3) is actually carried on the evolution equation (9.1 directly). The goal is to determine the “bulk burning rate”.

$$V(t) = \frac{1}{T} \int_0^T dt \int_{-h}^h \frac{1}{h} dx_2 \int_{-\infty}^{+\infty} u_t(t, x) dx_1$$

When u is a PTF, this rate converges to its average speed. Constantin, Kiselev, Oberman and Ryzhik carry the proofs in [22], [23] of 9.3 by developing some deep estimates on the evolution equation 9.1.

It is an open problem to determine rigorously the exact behavior of $c(A)$ and $c^*(A)$ as $A \rightarrow \infty$.

Related results have been obtained by Avellaneda and Majda [4], [5], [6] and Majda and McLaughlin [48], for turbulent advection-diffusion equations (with no reaction) with statistical velocity fields involving different spatial and temporal scales. They established bounds for the effective diffusivities of the renormalized equations at large spatio-temporal scales. The influence of advection on the effective diffusivity for advection-diffusion equations was

also dealt with by Fannjiang and Papanicolaou [25].

Other limits, related to homogenization with various scales for reaction-diffusion equations with advection, were considered by Majda and Souganidis [49], [50]. They also obtained some results on the asymptotic speed of propagation for the solutions of the Cauchy problem with very small diffusion and large reaction. Many interesting results regarding homogenization are also to be found in the works of Heinze [37], He3.

10 Estimates on the speed of propagation

To conclude this survey, I indicate three results which are taken from joint work with F. Hamel and N. Nadirashvili [11] and which are consequences of Theorem 8.3. They concern comparison results with the critical speed of planar fronts. Several more estimates can be found in [11].

It is assumed here that f is of the KPP type. To start with, consider the homogeneous equation (7.1) in a periodic domain Ω .

Theorem 10.1 [11] *Let $\vec{e}$ denote an arbitrary direction. The minimal speed c^* of pulsating fronts in direction $-\vec{e}$ for the homogeneous equation (7.1) in a periodic domain Ω , is never larger than the minimal speed of planar fronts $2\sqrt{f'(0)}$. Equality holds if and only if the domain Ω is invariant in the direction $\vec{e}$.*

Next, here is a property related to advection.

Theorem 10.2 [11] *Assume conditions (8.1)-8.3) on the field $\vec{q}$. In a straight infinite cylinder, $\Omega = \mathbb{R} \times \omega$, or in the whole space $\Omega = \mathbb{R}^n$, the minimal speed of pulsating fronts in the direction x_1 for the equation*

$$u_t - \Delta u + \vec{q}(x) \cdot \vec{\nabla} u = f(u),$$

is never smaller than the minimal speed with $q = 0$, namely $2\sqrt{f'(0)}$. Equality holds if and only if $\vec{q} \equiv 0$.

Hence, it is seen that while perforations slow down the asymptotic spreading, any stirring has the opposite effect of speeding it up.

Lastly, Section 5 dealt with the case of advection by a large shear flow, say in a straight infinite cylinder. It was proved in particular that the minimal

speed of traveling fronts behaves like the amplitude of the shear flow for large flow. One can ask whether this property holds for any periodic flow for the more general equation (7.9) (with $A \vec{q}(x)$) in the periodic setting. The answer is the following :

Theorem 10.3 [11] *The minimal speed $c^*(A)$ of the pulsating travelling fronts in the direction $-\vec{e}$ is such that*

$$\liminf_{A \rightarrow +\infty} c^*(A)/A > 0$$

if and only there exists a nonzero periodic function w , in $H_{loc}^1(\Omega)$, such that $\vec{q} \cdot \nabla w = 0$ almost everywhere, and $\int_C (\vec{q} \cdot \vec{e}) w^2 < 0$.

A particular consequence of this result concerns the case of a velocity field $\vec{q}$ considered in the previous section:

$$q_1 = -\frac{\partial \psi}{\partial x_2}, \quad q_2 = \frac{\partial \psi}{\partial x_1}.$$

Assume, say, that $\psi(x_1, x_2) = \cos(x_1) \cos(x_2)$, then it follows from the last theorem that

$$c^*(A) = o(A) \quad \text{as} \quad A \rightarrow +\infty.$$

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